NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE

SENIOR B DIVISION CONTEST NUMBER ONE FALL, 1988

Part I: 10 Minutes

F88B1. A parallelogram with two adjacent sides of 5 and 7 has the same <u>perimeter</u> as a square with side S. Compute S.

F88B2. If $x + y = 12$ and $x^2 - y^2 = 48$, compute x.

Part II: 10 Minutes NYCIML CONTEST ONE FALL, 1988

F88B3. In the Bronx Zoo, the ratio of snakes to lizards is 3:2, and the ratio of lizards to monkeys is 5:3. Compute the ratio of snakes to monkeys.

F88B4. If $\sin x = 3/5$, compute the numerical value of $\cos 2x$.

Part III: 10 Minutes NYCIML CONTEST ONE FALL, 1988

F88B5. In a sequence of numbers, the first number is 2 and the second number is 1. Each term after the second is formed by adding together all the previous terms of the sequence. What is the ninth term of the sequence?

F88B6. If $\log_8 12 = b$, express in terms of b the value of $\log_8 3$.

ANSWERS

1. 6
2. 8
3. 5:2 or equivalent
4. 7/25 or .28 or equivalent
5. 192
6. (3b-2)/3 or equivalent

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE
SENIOR B DIVISION CONTEST NUMBER TWO FALL, 1988

Part I: 10 Minutes

F88B7. If, for all real numbers x, $f(x) = x/2 - 3$, compute the numerical value of $f(f(f(8)))$.

F88B8. Compute the numerical value of

$$\left(\sqrt[3]{\sqrt{30} + \sqrt{3}}\right)\left(\sqrt[3]{\sqrt{30} - \sqrt{3}}\right).$$

Part II: 10 Minutes NYCIML CONTEST TWO FALL, 1988

F88B9. Find all real numbers x such that $x + \sqrt{x} = 12$.

F88B10. On a rectangular coordinate system, The point P has coordinates (4,2), and the point Q has coordinates (3,1). Find the coordinates of either of the two points on the line $y = x$ which is twice as far from point P as point Q is from point P.

Part III: 10 Minutes NYCIML CONTEST TWO FALL, 1988

F88B11. A bisector of an interior angle of a rectangle divides the side it intersects into two segments of length 12 and 6. Find the largest possible value for the <u>perimeter</u> of the rectangle.

F88B12. Compute the numerical value of

$$\log 1/2 + \log 2/3 + \log 3/4 + \log 4/5 + \ldots + \log 99/100,$$

where the base of each logarithm is 10.

ANSWERS

7. −4.25 or equivalent

8. 3

9. 9

10. $(3 + \sqrt{3}, 3 + \sqrt{3})$ or $(3 - \sqrt{3}, 3 - \sqrt{3})$ see solution for required form of answer.

11. 60

12. −2

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE

SENIOR B DIVISION CONTEST NUMBER THREE FALL, 1988

Part I: 10 Minutes

F88B13. Compute the numerical value of $.25^{0.5} + 0.0001^{0.25}$.

F88B14. The median of a trapezoid is 12 units long. It is divided by one of the diagonals of the figure into two segments, one of which is twice as large as the other. Find the length of the larger base of the trapezoid.

Part II: 10 Minutes NYCIML CONTEST THREE FALL, 1988

F88B15. If r and s are the roots of $x^2 - 7x + 11 = 0$,

compute the numerical value of $1/r^2 + 1/s^2$.

F88B16. A median of a triangle is equal in length to the geometric mean of the lengths of the sides that include it. If these two sides are 7 and 10, find the length of the side of the triangle to which the median is drawn.

Part III: 10 Minutes NYCIML CONTEST THREE FALL, 1988

F88B17. The sides of a parallelogram are 18 inches and 12 inches. The smaller altitude of the parallelogram is 9 inches. Find the inch-length of the longer altitude of the parallelogram.

F88B18. A train to Albany was delayed at Grand Central Station for six minutes. The engineer made up for the delay by increasing her speed by 4 miles per hour for a distance of 36 miles, and then resuming her normal speed. What was this normal speed, in miles per hour?

ANSWERS

13. 0.6 or equivalent

14. 16

15. 27/121 or equivalent

16. $3\sqrt{2}$ or equivalent

17. 13.5 or 27/2 [inches]

18. 36 [miles per hour]

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE
SENIOR B DIVISION CONTEST NUMBER FOUR FALL, 1988

Part I: 10 Minutes

F88B19. The n^{th} term in the sequence a_n is defined by

$a_n = 1 + (-1)^{n+1}$. Find the sum of the first twenty terms of this sequence.

F88B20. A regular dodecagon (regular 12-gon) is inscribed in a circle of radius 12. Compute the area of the dodecagon.

Part II: 10 Minutes NYCIML CONTEST FOUR FALL, 1988

F88B21. Find a real value of a so that the simultaneous equations $x + 3y = 7$,
$3x - ay = 21$ have infinitely many solutions in common (for x and y).

F88B22. In right triangle ABC, CM is the median to hypotenuse AB. IF angle A measures 60 degrees and AB = 12, compute the distance from point B to line CM.

Part III: 10 Minutes NYCIML CONTEST FOUR FALL, 1988

F88B23. If $\sin^2 x - \cos^2 x = 2/3$, compute the numerical

value of $\sin^4 x - \cos^4 x$.

F88B24. Two circles are tangent externally, and the sum of their areas is 130π square inches. If the distance between their centers is 14 inches, find the number of inches in the length of the radius of the <u>larger</u> of the two circles.

ANSWERS

19. 20 21. -9 23. $2/3$

20. 432 22. $3\sqrt{3}$ 24. 11

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE

SENIOR B DIVISION CONTEST NUMBER FIVE FALL, 1988

Part I: 10 Minutes

F88B25. If $z = 8$, compute the numerical value of

$$\left(z^{4/9} \right)^{-3} \left(z^{1/6} \right)^{-2}$$

F88B26. Find the smallest integer value of x for which $(x-3)(x-7) \leq 5(x-2)$.

Part II: 10 Minutes NYCIML CONTEST FIVE FALL, 1988

F88B27. For all real numbers x and y, with $y \neq 0$, the function $f(x)$ satisfies the condition $f(x/y) = f(x) + f(y)$. Compute the numerical value of $f(11)$.

F88B28. Line XY is tangent to circle O (with center O) at point X and to circle P (with center P) at point Y. The radii of the circles are 5, and 8, respectively, and points O and P are both on the same side of line XY. If $XY = \sqrt{7}$, compute the length of OP.

Part III: 10 Minutes NYCIML CONTEST FIVE FALL, 1988

F88B29. Ten percent of 9 is 9 percent of what number?

F88B30. In a rectangular coordinate system, a tangent from the point (24,7) to the circle whose equation is

$x^2 + y^2 = 400$ has a point of tangency at (a,b), where b is a positive real number. Compute a.

ANSWERS

25. 1/32, or equivalent

26. 3

27. 0

28. 4

29. 10

30. 12

SOLUTIONS

F88B1. Since opposite sides of a parallelogram are equal, the perimeter of the parallelogram is $5+7+5+7 = 24$. Since the four sides of a square are equal, the side of the square is $24/4 = 6$.

F88B2. Using the fact that $x^2 - y^2 = (x+y)(x-y)$, we can write $48 = 12(x-y)$, or $x-y = 4$. Now we have the simultaneous equations $x+y = 12$, $x - y = 4$, which have the solution $x = 8$, $y = 4$.

F88B3. The ratio of snakes to lizards can be written as 15:10, while the ratio of lizards to monkeys can be written as 10:6. From this it is clear that the ratio of snakes to monkeys is 15:6 or 5:2.

F88B4. We have $\cos 2x = 1 - 2 \sin^2 x = 1 - 2(9/25) = 1 - 18/25 = 7/25$.

F88B5. The sequence begins: 2, 1, 3, 6, 12, 24, 48, 96, 192, and each successive term is easily seen to be twice the previous term. Hence the sum of the first eight terms is simply the ninth term, which is 192.

F88B6. We have $\log_8 2 = 1/3$,

$$\log_8 4 = \log_8 2^2 = 2 \log_8 2 = 2/3,$$

$$\log_8 3 = \log_8 12/4 = \log_8 12 - \log_8 4 =$$

$$= b - 2/3 \text{ or } (3b-2)/3.$$

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE
SENIOR B DIVISION CONTEST NUMBER TWO FALL, 1988

SOLUTIONS

F88B7. If $f(x) = x/2 - 3$, $f(8) = 4-3 = 1$, $f(f(8)) = f(1) = 1/2 - 3 = -2.5$, and $f(f(f(8))) = -1.25 - 3 = -4.25$ or $-17/4$.

F88B8. We have $(\sqrt[3]{\sqrt{30} + \sqrt{3}})(\sqrt[3]{\sqrt{30} - \sqrt{3}}) =$

$$\sqrt[3]{(\sqrt{30} + \sqrt{3})(\sqrt{30} - \sqrt{3})} = \sqrt[3]{30 - 3} = \sqrt[3]{27} = 3.$$

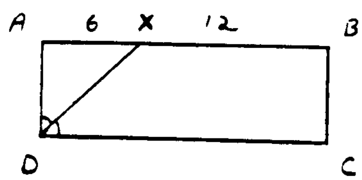
F88B9. Letting $y = \sqrt{x}$, we have $y^2 + y = 12$, or $y^2 + y - 12 = (y+4)(y-3) = 0$, from which $y = 3, -4$. Then $\sqrt{x} = 3$ or -4, and the second value is extraneous. We are left with $\sqrt{x} = 3$, or $x = 9$.

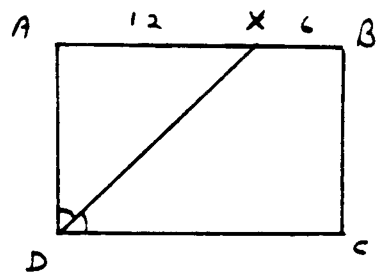
F88B10. The distance between points P and Q is $\sqrt{2}$ (a diagram shows this quickly). Hence we need a point whose coordinates are equal, and whose distance to the point $(4,2)$ is $2\sqrt{2}$. If the coordinates of such a point are (c,c), then

we have $(c-4)^2 + (c-2)^2 = (2\sqrt{2})^2 = 8$, or $c^2 - 6c + 6 = 0$. The roots of this equation are $3 + \sqrt{3}$ and $3 - \sqrt{3}$, so the two points are either $(3 + \sqrt{3}, 3 + \sqrt{3})$ or $(3 - \sqrt{3}, 3 - \sqrt{3})$.

Either answer is acceptable, so long as it is written as an ordered pair.

F88B11. There are two possibilities, indicated by the following two diagrams:





In either case, triangle ADX is isosceles, so that we can compute the smaller side. The perimeter of the first rectangle is 48, while that of the second rectangle is 60. The largest possible value, then, is 60.

F88B12. The given expression is equal to

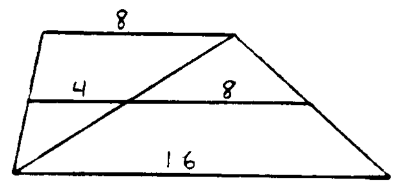
$$\log 1 - \log 2 + \log 2 - \log 3 + \log 3 - \log 4 + \dots + \log 99 - \log 100.$$

This sum "telescopes": adjacent terms cancel out, to give a final sum of $\log 1 - \log 100$. Since the base of each logarithm is 10, this expression is equal to $0 - 2 = -2$.

SOLUTIONS

F88B13. We have $.25^{0.5} = \sqrt{.25} = .5$, while $0.0001^{0.25} =$ $= \sqrt[4]{0.0001} = 0.1$. The sum of these two numbers is 0.6 or .6.

F88B14. The diagonal of a trapezoid divides the figure into two triangles. The segment of the median of the trapezoid which is contained in each triangle is the segment connecting the midpoints of two sides of the triangle. This segment is thus half the third side (and parallel to the third side). Since the segments here are 8 and 4, the larger base is 16.

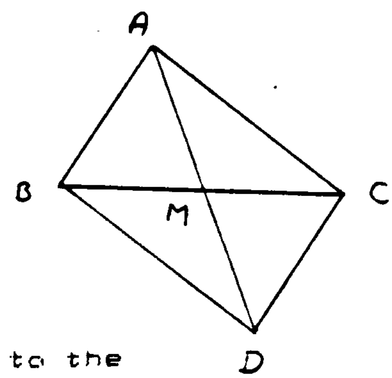


F88B15. We have $1/r^2 + 1/s^2 = (r^2 + s^2)/r^2s^2$. Now $r^2s^2 =$ $11^2 = 121$, while $r^2 + s^2 = (r+s)^2 - 2rs = (+7)^2 - 2\cdot 11 =$ $= 49 - 22 = 27$. The required value is thus 27/121.

F88B16. If we extend AM its own length to D (see diagram), then $AM = \sqrt{70}$, so $AD = 2\sqrt{70}$. Then ACDB is a parallelogram, so the sum of the squares of the sides equals the sum of the squares of its diagonals, or

$AD^2 + BC^2 = 2AB^2 + 2AC^2$. If $BC = x$, then $x^2 + 280 = 298$, and $BC = \sqrt{18} = 3\sqrt{2}$.

An alternate solution might use the law of cosines in triangles ABC and ABD, solving for sides AD and BC. Since angles BAC and ABD are supplementary, their cosines are opposite in sign but equal in absolute value.



F88B17. The area of the parallelogram is equal to the product of any side and the altitude to that side. It follows that the shorter altitude corresponds to the longer side, and the area of the given parallelogram is $9\cdot 18 = 162$ square inches. Hence the longer altitude is $162/12 = 13.5$ inches, or 27/2 inches.

F88B18. If the normal speed was x (in miles per hour), then the "normal time" in is $36/x$ hours, and the time the train actually took is $36/(x+4) + 1/10$ hours (since 6 minutes is 1/10 hour). These two times must be equal, so that

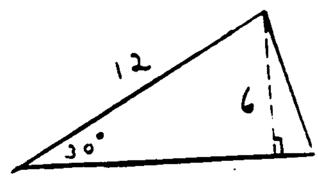
$36/x = 36/(x+4) + 1/10$. Solving for x, we have:

$360(x+4) = 360x + x(x+4)$, or $1440 = x^2 + 4x$, or $x^2 + 4x - 1440$ $= (x + 40)(x - 36) = 0$, so that $x = -40$ or $x = 36$. The first solution is extraneous.

SOLUTIONS

F88B19. By computing the first few terms of the sequence, it is easy to see that the even numbered terms are zero and the odd numbered terms are 2. The sum of the first twenty terms is thus 20.

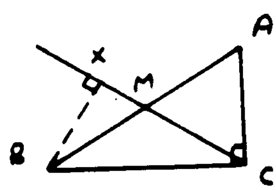
F88B20. If we "slice" the dodecagon into twelve isosceles triangles, we find that the vertex angle of each isosceles triangle is 30 degrees (see diagram).



We can find the area of each "slice" by drawing an altitude to the leg of the isosceles triangle. This forms a 30-60-90 triangle, whose hypotenuse is a radius of the circle, or 12. Thus the altitude is half the hypotenuse, or 6, the area of the triangular slice is $(1/2)(12)(6) = 36$, and the area of the dodecagon is $12 \cdot 36 = 432$.

F88B21. Multiplying the first equation by 3 gives $3x + 9y = 21$. If we choose $a = -9$, the second equation is identical to this, and the two equations have infinitely many solutions in common. It can be shown that this is the only such value for a.

F88B22. The median to the hypotenuse of a right triangle is equal to half the hypotenuse. Therefore (see diagram), angle MCB = angle ABC = 30 degrees, and triangle BXC is a 30-60-90 triangle, so $BX = BC/2 = (1/2)(\sqrt{3}/2)(AB) = 3\sqrt{3}$.



F88B23. Factoring, we have $\sin^4 x - \cos^4 x =$

$(\sin^2 x + \cos^2 x)(\sin^2 x - \cos^2 x) = 1 \cdot (\sin^2 x - \cos^2 x) = 2/3$.

F88B24. If the radii of the two circles are r and s, then the distance between their centers is $r + s = 14$. Since the

sum of the areas is 130π, $\pi r^2 + \pi s^2 = 130\pi$, so $r^2 + s^2 = 130$. Squaring the first equation gives $r^2 + s^2 + 2rs = 196$, and subtracting the second equation gives $2rs = 66$, or $rs = 33$. Since $r + s = 14$, r and s are roots of the equation $x^2 - 14x + 33 = (x - 11)(x - 3) = 0$, so $x = 11$ or 3. Thus r and s are 11 and 3 (in some order), and the radius of the larger circle is 11.

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE
SENIOR B DIVISION CONTEST NUMBER FIVE FALL, 1988

SOLUTIONS

F88B25. We have:

$$(z^{4/9})^{-3}(z^{1/6})^{-2} = z^{-4/3}z^{-1/3} = z^{-5/3}.$$ Since $8^{1/3} = 2$, $8^{-5/3} = 2^{-5} = 1/32$.

F88B26. We have: $x^2 - 10x + 21 \le 5x - 10$
$x^2 - 15x + 31 \le 0$

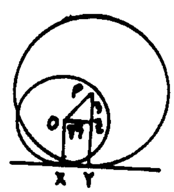
If we look at the graph of the function $y = x^2 - 15x + 31$, we find that it is a parabola opening upward. The quadratic formula gives its two roots as

$x = (15 \pm \sqrt{101})/2$, or approximately $(15 \pm 10)/2$, which are two numbers very close to 25/2, 5/2. The shape of the parabola tells us that the function is negative <u>between</u> the two roots, and positive outside. The smallest integral value lying between the two roots is 3.

F88B27. If we let $x = 0$, then the given condition on f reads $f(0/y) = f(0) + f(y)$, or $f(0) = f(0) + f(y)$. Subtracting $f(0)$ from each side (whatever value it may have) gives $f(y) = 0$. Since this is true for all $y \ne 0$ (in particular, for $y = 11$), we must have $f(11) = 0$.

Must $f(0) = 0$ as well? What can you say about $f(x)$ if x is <u>not</u> an integer? What kind of function is this, anyway?

F88B28. It is clear that OX and PY are both perpendicular to XY. If we draw OZ parallel to XY (see diagram), then, in triangle OZP, $OP^2 = OZ^2 + PZ^2 = 7 + (PY - ZY)^2 = 7 + (PY - OX)^2 = 16$. Then $OP = 4$.



F88B29. Ten percent of 9 is $(1/10)(9) = 9/10$. Then, if x is the required number, $9x/100 = 9/10$, so $x = 10$.

Trying a related problem with the number 9 replaced by 90 will give some insight into why this works.

F88B30. Since OT is a radius of the given circle (see diagram), $OT = \sqrt{400} = 20$. Since $OQ = 24$,, and $PQ = 7$, an application of the Pythagorean theorem to triangle OPQ shows that $OP = 25$. Then, if angle $POQ = c$, and angle $TOP = d$, we can write angle $TOQ = c + d$, and $OX = a = OT \cos(c+d)$. Now $\cos c = OQ/OP = 24/25$, $\cos d = OT/OP = 20/25$, and it is not hard to see then that $\sin c = 7/25$, $\sin d = 15/25$
Hence $\cos(c+d) = (24/25)(20/25) - (7/25)(15/25) = 3/5$, and $OX = 20(3/5) = 12$.