New York City
Interscholastic Mathematics League

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE

SENIOR DIVISION CONTEST NUMBER ONE FALL, 1988

Part I: 10 Minutes

F88S1. Find two positive integers whose product is 24999999, and whose (positive) difference is as small as possible.

F88S2. Compute $[\sqrt[3]{(\sqrt{30}+\sqrt{3})}]\,[\sqrt[3]{(\sqrt{30}-\sqrt{3})}]$

Part II: 10 Minutes NYCIML CONTEST ONE FALL, 1988

F88S3 If i denotes the imaginary unit, compute the product

$i\cdot i^2\cdot i^3\cdot i^4\cdot i^5\cdot i^6\cdot \;...\; \cdot i^{98}\cdot i^{99}\cdot i^{100}$.

F88S4. Fixed points A and B are chosen on circle O. They are the endpoints of one of the legs (one of the non-parallel sides) of a trapezoid inscribed in the circle. For all such trapezoids, the set of points of intersection of diagonals AC and BD forms a finite curve. If AB = 12 and the radius of circle O is also 12, find the length of this curve.

Part III: 10 Minutes NYCIML CONTEST ONE FALL, 1988

F88S5. Points A, B, C, and D lie on the same line, in the order mentioned. If AB:AC = 1:3, and BC:BD = 4:5, compute the ratio AB:CD.

F88S6. How many decimal numerals are there which are made up of the digits 1, 2, 3, 4, 5, each used at most once, and which represent multiples of 8?

ANSWERS

1. 4999, 5001
2. 3
3. -1
4. Ans: $16\pi\sqrt{3}/3$.
5. 2 or 2:1
6. 27

New York City
Interscholastic Mathematics League

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE
SENIOR DIVISION CONTEST NUMBER TWO FALL, 1988

Part I: 10 Minutes

F88S7. Tubby can eat 3 pies in 4 minutes. At this rate, in how many minutes can he eat 5 pies?

F88S8. Regular octagon ABCDEFGH, with its vertices labelled in clockwise order, is drawn on a rectangular coordinate plane. If the coordinates of A are (4,0) and the coordinates of B are (0,4), If the coordinates of vertex E are (p,q), compute q - p.

Part II: 10 Minutes NYCIML CONTEST TWO FALL, 1988

F88S9. For all positive real numbers x, let $f(x) = \log_x 10$. Find all real numbers x such that $f(x) = 1/2$.

F88S10. The cube of a two-digit number (in decimal notation), whose digits are all different, is a five digit number, whose digits are all different, and are also different from those of the original number. Find the original number.

Part III: 10 Minutes NYCIML CONTEST TWO FALL, 1988

F88S11. How many arithmetic progressions are there, with at least three terms, whose terms are all positive integers, whose first term is 3, and whose last term is 21?

F88S12. If a, b, and c are positive real numbers such that abc = 1, find the minimal possible value of (a+1)(b+1)(c+1).

ANSWERS

7. 20/3 or equivalent	9. 100	11. 5
8. 4	10. 27	12. 8

New York City
Interscholastic Mathematics League

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE
SENIOR DIVISION CONTEST NUMBER THREE FALL, 1988

Part I: 10 Minutes

F88S13. The first term of a geometric progression is $\sqrt{2}$, and the second term is $\sqrt{3}$. Compute the fifth term.

F88S14. In a certain code, each of the 26 letters of the alphabet is represented by the number which is its usual alphabetic order. For instance, A is represented by 1, B by 2, . . . , and Z by 26. A word is then encoded by multiplying together the numbers which represent its letters. What English word is encoded as 120175?

Part II: 10 Minutes NYCIML CONTEST THREE FALL, 1988

F88S15. The length of a rectangle is π, and its width is r. Express in terms of r the radius of a circle whose area is equal to that of the rectangle described.

F88S16. For all real numbers x, find the largest possible value of the expression sin x + cos x.

Part III: 10 Minutes NYCIML CONTEST THREE FALL, 1988

F88S17. Find all positive real numbers x such that

$$[\log x^2]^2 = \log 10000,$$

where the base of each logarithm is 10.

F88S18. Point M is the centroid (intersection of the medians) of right triangle ABC, (with $AC \perp BC$). Points X, Y, and Z are the feet of the perpendiculars from M to AB, BC, AC respectively. Find the area of triangle XYZ if AC = 3 and BC = 12.

ANSWERS

13. $9\sqrt{2}/4$

14. WEEKS

15. $\sqrt{r}$

16. $\sqrt{2}$.

17. 10, 1/10

18. 4

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE
SENIOR DIVISION CONTEST NUMBER FOUR FALL, 1988

Part I: 10 Minutes

F88S19. The first two terms of an arithmetic progression are $\log_2 3$ and $\log_2 9$. If the sixth term is x, compute the numerical value of 2^x.

F88S20. If f(x), defined for all real x, satisfies $f(1-x) + 2f(x) = x$, express f(x) as a polynomial in x.

Part II: 10 Minutes NYCIML CONTEST FOUR FALL, 1988

F88S21 Two identical oil pipelines are each circular cylinders of radius 6. The Environmental Protection Agency has ordered that they be replaced with a single pipeline with the same capacity. If the new pipeline is also to be a circular cylinder, what must its radius be?

F88S22. For all real numbers x, find the largest possible value of the expression 2sin x + cos x.

Part III: 10 Minutes NYCIML CONTEST FOUR FALL, 1988

F88S23. A student wrote the algebraic expression

(((x - 7)^((x+3)-(x+4)^2) - (x - 3)^(x^(2x)+1)(x^2+1),

in which the symbol "^" indicates raising to a power. The student left out several closing parentheses. How many?

F88S24. The area of triangle ABC is equal to $BC^2 + AC^2 - AB^2$. If angle C is acute, compute the numerical value of its secant.

ANSWERS

19. 729.
20. $x - 1/3$ or $(3x - 1)/3$ or equivalent.
21. $6\sqrt{2}$
22. $\sqrt{5}$.
23. 2
24. $\sqrt{17}$.

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE
SENIOR DIVISION CONTEST NUMBER FIVE FALL, 1988

Part I: 10 Minutes

F88S25. The number 33 is a multiple of 11, and leaves a remainder of 1 upon division by 4. Find the next largest integer which is a multiple of 11, and which leaves a remainder of 1 upon division by 4.

F88S26. For all real x, $f(x) = x + 3$, and $g(x)$ is a polynomial of degree 2 such that $g(f(x)) = x^2 + 2$. Find $g(x)$.

Part II: 10 Minutes NYCIML CONTEST FIVE FALL, 1988

F88S27. A pair of two-dimensional beings play two-dimensional darts. Their dart board is the interval $0 \leq x \leq 1$ on a number line. Their darts land at points on this interval. They never miss the interval. What is the probability that the first dart thrown will end up between the second dart and the third dart?

F88S28. For all real numbers x, find the smallest possible positive value of the expression tan x + cot x.

Part III: 10 Minutes NYCIML CONTEST FIVE FALL, 1988

F88S29. In isosceles triangle ABC, BT is the bisector of base angle ABC (with point T on leg AC). If $m\angle BTC = 135$, $BT = 6$, find BC.

F88S30. If [x] denotes the greatest integer not exceeding x, compute the numerical value of

$$\sum_{n=0}^{\infty} [(10000 + 2^n)/2^{n+1}] =$$

$$= [(10000+1)/2] + [(10000+2)/4] + [(10000+4)/8] + [(10000+8)/16] + \ldots$$

ANSWERS

25. 77

26. $x^2 - 6x + 11$

27. 1/3

28. 2

29. $6\sqrt{2}$.

30. 10000

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE
SENIOR DIVISION CONTEST NUMBER ONE FALL, 1988

SOLUTIONS

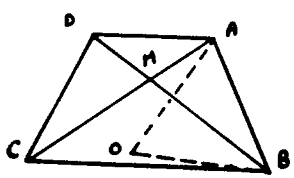
F88S1. It is not difficult to see that 24999999 was chosen because of its proximity to the square 25000000. In fact,

$24999999 = 25000000 - 1 = 5000^2 - 1^2 = (5000 + 1)(5000 - 1)$ $= 4999 \cdot 5001$. Note that only the first of these two factors is prime.

F88S2. We have: $[\sqrt[3]{(\sqrt{30} + \sqrt{3})}]\,[\sqrt[3]{(\sqrt{30} - \sqrt{3})}] = \sqrt[3]{(30 - 3)}$ $= \sqrt[3]{27} = 3$.

F88S3. Since $i \cdot i^2 \cdot i^3 \cdot i^4 = (i)(-1)(-i)(1) = i^2 = -1$, each set of four factors in the product equals -1. Hence the product up to i^{100} equals $-1^{25} = -1$.

F88S4.



The required length is that of the locus of point M (see diagram). The degree measure of angle AMB is equal to the average of minor arcs AB, CD. Since the trapezoid is isosceles, this is equal to the measure of arc AB. Since the chord of this arc is equal to a radius of the circle, the arc is 60°, and the required locus is 2/3 of the circle circumscribing triangle ABO. This triangle is equilateral, and its circumradius is 2/3 of its altitude, or $4\sqrt{3}$. The length of the required arc is then $(2/3)(8\pi\sqrt{3}) = 16\pi\sqrt{3}/2$.

F88S5.

We have AB:BC = 1:2, So AB = (1/2)BC. It is given that CD = (1/4)BC, so AB:BC = (1/2):(1/4) or 2.

F88S6. Clearly there are no such single-digit numbers. Suppose such a number has two digits. Then it must be even (since 8 is even), so the units digit is either 2 or 4. If the units digit is 2, a quick check shows that only 32 works. If the units digit is 4, only 24 works. There are two such numerals which contain two digits.

Suppose such a number has three digits. A multiple of 8 must be a multiple of 4. We can check for multiples of 4 by looking at the rightmost two digits: if they form a multiple of 4, the entire number is a multiple of 4. This requires that the rightmost two digits be 12, 24, 32, or 52. Taking each case separately, we find that there are only the cases 312, 512, 432, 152, and 352. There are five such numerals which contain three digits.

Now it is easy to count the four- or five-digit numerals. Since 1000 is a multiple of 8, any four- or five-digit number ending in 312 is of the form 1000N + 312, and hence is a multiple of 8. There are four such numerals which fit our problem: 4312, 5312, 45312, 54312. A similar pattern holds for each of the three-digit numerals already found, which makes $4 \cdot 5 = 20$ four- or five- digit numerals possible.

Altogether, there are 2 + 5 + 20 = 27 such numerals.

For more information on criteria for divisibility, see any book on number theory.

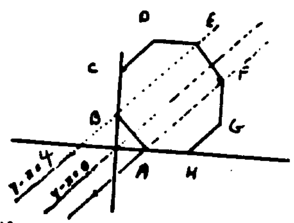
NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE
SENIOR DIVISION   CONTEST NUMBER TWO   FALL, 1988

SOLUTIONS

F88S7. Tubby consumes pies at the rate of 1 every 4/3 minute, so it takes him 20/3 minutes to consume 5 pies.

F88S8. Method I: direct computation of p and q.

Method II: The line parallel to the line $x = y$ (or $y - x = 0$) and passing through point E has an equation $y - x = b$, where b is the y-intercept of the line. A construction of this line (see diagram) shows that $b = 4$



F88S9. Solution. Since $\log_x 10 = 1/(\log_{10} x)$, we must have $\log_{10} x = 2$, so that $x = 100$.

F88S10. Suppose $XY^3 = ABCDE$. Since $Y \neq E$, Y must equal 2, 3, 7, or 8. We can estimate the size of xy by noting that

$21^3 = 9261 < 10\,000 < XY^3 < 100\,000 < 103823 = 47^3$, so that XY can equal only 23, 27, 28, 32, 37, 38, 42, 43. By trial and error, $27^3 = 19683$ works, and it is the only such number.

F88S11. If the n^{th} term is 21, and the common difference is d, then $a_n = a_1 + (n-1)d$

$$21 = 3 + (n-1)d,$$

so $d(n-1) = 18$, and we can obtain possible values of n and d by factoring 18. We find that d = 1, 2, 3, 6, 9, 18, with corresponding values for n of 19, 10, 7, 4, 3, 2 respectively. Only the first five values of n satisfy the requirements of the problem.

F88S12. Let $M=(a+1)(b+1)(c+1)$. Then $M/1 = (a+1)(b+1)(c+1)/abc = \frac{a+1}{a} \cdot \frac{b+1}{b} \cdot \frac{c+1}{c}$

$$= (1 + 1/a)(1 + 1/b)(1 + /1c).$$

Multiplying these two expressions for M, we find tha

$$M^2 = (1+a)(1+b)(1+c)(1 + 1/a)(1 + 1/b)(1 + /1c),$$

$$= (2 + a + 1/a)(2 + b + 1/b)(2 + c + 1/c).$$

Now for $x > 0$, $x + 1/x \geqslant 2$ with equality when $x = 1$.

Hence the minimal value for M^2 occurs when $a = b = c = 1$,

and is equal to 4^3. This means that the minimal value for M is 8.

To show quickly that $x + 1/x$ achieves a minimum when $x = 1$ (without calculus!), we can use the "obvious" inequality

$(\sqrt{x} - \sqrt{(1/x)})^2 \geqslant 0$, which is true since the square of a real number is never negative.

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE
SENIOR DIVISION CONTEST NUMBER THREE FALL, 1988

SOLUTIONS

F88S13. The common ratio is $\sqrt{(3/2)}$, so the fifth term is $(\sqrt{2})(\sqrt{3/2})^4 = (9/4)\sqrt{2}$.

F88S14. We must factor the number 120175. Clearly, this number is a multiple of 25 and of 11, and with these clues

it is not hard to see that $120175 = 5^2 \cdot 11 \cdot 19 \cdot 23$. It remains to decode the word uniquely. The numbers 11, 19 and 23, being prime, can only represent K, S and W respectively.

The factor 5^2 could represent the letter Y, or it could represent two copies of the letter E.

We are left with the combinations WYSK or SEEKW. Only the second can be anagrammed into an English word: the word WEEKS.

F88S15. If the radius of the required circle is x, then $\pi x^2 = \pi r$, so $x^2 = r$, and $x = \sqrt{r}$.

F88S16. If $M = \sin x + \cos x$, then $M^2 =$

$\sin^2 x + \cos^2 x + 2 \sin x \cos x = 1 + 2 \sin x \cos x = 1 + \sin 2x$. This has its maximal value when $\sin 2x = 1$, which gives $\sqrt{2}$ as a maximum for M.

The same answer can be achieved by writing M as $(\sqrt{2})(1/\sqrt{2} \sin x + 1/\sqrt{2} \cos x) = (\sqrt{2})(\cos \pi/2 \sin x + \sin \pi/4 \cos x) = \sqrt{2} \sin (x + \pi/4)$. Since the sine of an angle cannot exceed 1, this again gives $\sqrt{2}$ as a maximum.

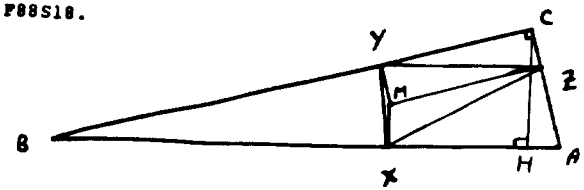
F88S17. We have

$$[\log x^2]^2 = \log 10000 = \log 10^4 = 4,$$

so $\log x^2 = 2 \log x = 2$ or -2, and $\log x = 1$ or -1. This shows that $x = 10, 1/10$.

Note that there are two more answers which are negative real numbers.

F88S18.



The Pythagorean theorem shows that $AB = 3\sqrt{17}$. If CH is the altitude from C to AB, then (by computing the area of triangle ABC in two different ways, or otherwise), $CH = 12/\sqrt{17}$.

We will need the following lemma:

The distance from the centroid of a triangle to any side is one-third the altitude to that side.

The proof follows from the fact that the medians of a triangle trisect each other, and from noting that the perpendiculars from the centroid to each side are parallel to the altitude to that side (see diagram).

In the present case, $MX = CH/3 = 4/\sqrt{17}$, $MY = AC/3 = 1$, and $MZ = BC/3 = 4$ (since either leg of a right triangle is the altitude to the other leg).

We now find the area of triangle XYZ by adding the areas of triangles MXY, MYZ, and MXY. We can compute the area of each of these smaller triangles using the formula $K = (1/2)ab \sin C$. The sine of each of the angles at M is equal to the sine of one of the angles of the original triangles, since these pairs of angles are supplementary.

$$\sin \angle ZMY = \sin \angle ACB = 1,$$
$$\sin \angle ZMX = \sin \angle CAB = 4/\sqrt{17},$$
$$\sin \angle YMX = \sin \angle CBA = 1/\sqrt{17}.$$

Then, using absolute value for area,

$$|ZMY| = (1/2)(4)(1)(1) = 2$$
$$|ZMX| = (1/2)(4)(4/\sqrt{17})(4/\sqrt{17}) = 32/17,$$
$$|XMY| = (1/2)(1)(4/\sqrt{17})(1/\sqrt{17}) = 2/17,$$

so that $|XYZ| = 2 + 2/17 + 32/17 = 2 + 34/17 = 4$.

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE
SENIOR DIVISION CONTEST NUMBER FOUR FALL, 1988

SOLUTIONS

F88S19. If the n^{th} term is a_n, then $a_6 = a_1 + 5d$, where d is the common difference. Here, $d = \log_2 9 - \log_2 3 = 2\log_2 3 - \log_2 3 = \log_2 3$, so $x = a_6 = 6\log_2 3 = \log_2 3^6$, and $2^x = 3^6 = 729$.

F88S20. We are given (i) $f(1-x) + 2f(x) = x$.
Replacing x with 1-x:

(ii) $f(x) + 2f(1-x) = 1-x$.

Adding: $3f(x) + 3f(1-x) = 1$,

or (iii) $f(x) + f(1-x) = 1/3$.

Subtracting (iii) from (ii):

$f(1-x) = 1 - x - 1/3 = 2/3 - x$,

so $f(x) = 2/3 - (1-x) = x - 1/3$.

F88S21. The capacity can be measured by the cross sectional area, since it is determined by the rate at which the oil passes through the pipe. The old cross sectional area is $2 \times 36\pi = 72\pi$. This must be π times the square of the new radius, which thus is $\sqrt{72}$ or $6\sqrt{2}$.

F88S22. If $M = 2 \sin x + \cos x$, then
$M/\sqrt{5} = 2/\sqrt{5} \sin x + 1/\sqrt{5} \cos x$. Now $(2/\sqrt{5})^2 + (1/\sqrt{5})^2 = 1$, so there is a number A such that $\cos A = 2/\sqrt{5}$ and $\sin A = 1/\sqrt{5}$. Then we can write

$M/\sqrt{5} = \cos A \sin x + \sin A \cos X$
$= \sin(x + A)$, for a certain angle A.

Hence $M/\sqrt{5}$ can be no larger than 1, and M achieves a maximal value of $\sqrt{5}$ when $x = \pi/2 - A$.

F88S23. We can "scan" the expression from left to right. Starting with a "parenthesis count" of zero, we add one for each opening parenthesis, and subtract one for each closing parenthesis. The result, if positive, will tell us how many closing parentheses are missing. The answer is 2.

Note that this does not tell us exactly what was intended for the meaning of the expression. This is a common difficulty in programming in the LISP computer language.

In the scan described above, what would a negative result tell us?

F88S24. The form of the given expression for area is reminiscent of the law of cosines. Letting AB = c, AC = b, BC = a, we have $(1/2)ab \sin C = a^2 + b^2 - c^2$

$= a^2 + b^2 - [a^2+b^2-2ab \cos C]$
$= 2ab \cos C$.

Hence $\tan C = (\sin C)/(\cos C) = 2/(1/2) = 4$,

$\sec^2 C = 1 + \tan^2 C = 17$, and $\sec C = \sqrt{17}$.

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE
SENIOR DIVISION CONTEST NUMBER FIVE FALL, 1988

SOLUTIONS

F88S25. The problem discusses multiples of 11 which are of the form $4n+1$; that is, they are congruent to 1 modulo 4. Since 11 is congruent to 3 modulo 4, we need to multiply 11 by another number congruent to 3 in order to get an answer which is congruent to 1. Thus the next such number is the product of 11 and the next number congruent to 3 modulo 4. This is $11 \cdot 7 = 77$.

The same answer can be arrived at by trial and error.

F88S26. $[f(x)]^2 = x^2 + 6x + 9$
$-6\ f(x) = \quad -6x - 18$

Adding, $[f(x)]^2 - 6\ [\ f(x)\ ] = x^2 - 9$. Adding 11, we find

$[f(x)]^2 - 6f(x) + 11 = x^2 + 2 = g(f(x))$, so $g(x) = x^2 - 6x + 11$.

F88S27. There is no reason that any one of the three darts should be more likely to land between the other two. Since one dart must land between the other two, the probability that this dart is the first (or the second, or the third), must be 1/3.

This problem can be visualized using volumes. We are seeking the volume of the region of space satisfying $0 \leq x < y < z \leq 1$, where x, y, and z are Cartesian coordinates, representing the points where the darts land.

This geometric interpretation explains why we have neglected throughout the vanishing probability that two darts land on the same point. This assumption does not affect the answer.

It is also possible, although not necessary, to use calculus to solve this problem. If the first lands at point x, then another dart will land to the left of x with probability x, and to the right of x with probability $1-x$. Thus the probability that the second and third darts surround a dart at point x is given by $2x(1-x)$. Integrating this expression as x runs from 0 to 1 gives the answer.

F88S28. Since $(\tan x)(\cot x) = 1$, the product of these two numbers is constant. It follows that their sum is minimal when they are equal. This occurs, for instance, when $x = \pi/4$, and the minimal value is 2.

F88S29. It follows that angle C measures 30°. Using the law of sines in triangle BTC, and letting $x = BC$, we have:

$$\frac{x}{\sin 135^\circ} = \frac{6}{\sin 30^\circ}$$

F88S30. First note that each summand can be written as

$$[\ 10000/2^{n+1} + 1/2\ ].$$

It follows that the given sum is not really infinite, since for $2^{n+1} > 10000$, the summand is zero. Hence we are concerned only with finitely many of these summands. But how many? And what do they equal?

An answer is obtained from the fact that, for a positive number x, $[x + 1/2] = [x]$ or $[x] + 1$, according as $x - [x]$ is greater or less than 1/2. In fact, if we check these two cases, we find that $[x + 1/2] = [2x] - [x]$ for any x at all.

Using this relationship, with $x = 10000/2^{n+1}$, we can rewrite the given sum as:

$$\sum_{n=0}^{\infty} [10000/2^n] - [10000/2^{n+1}].$$

This sum telescopes, and we find that the partial sum from $n = 0$ to $n = k$ (for any large integer k) is equal to

$10000 - [10000/2^k]$. For sufficiently large k, this is equal to 10000.

This problem can also be solved by direct, but tedious, computation.

January 29, 1989

Dear Math Team Coach,

Enclosed is your copy of the FALL, 1988 NYCIML contests that you requested on the application form.

The following are the corrected answers for the enclosed contests:

	Question	Correct answer
Senior A	F88S29	Add the following to the question: "BC is the base."

Have a great spring term!

Sincerely yours,

Richard Geller

Secretary, NYCIML