New York City
Interscholastic Mathematics League

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE
JUNIOR DIVISION CONTEST NUMBER ONE FALL, 1988

Part I: 10 Minutes

F88J1. A cubic foot is what fraction of a cubic yard?

F88J2. The decimal numeral 1287xy6, where x and y stand for decimal digits, represents a multiple of 72. For how many ordered pairs (x,y) is this true?

Part II: 10 Minutes NYCIML CONTEST ONE FALL, 1988

F88J3. If $P = 2^{1988} + 2^{-1988}$, and $Q = 2^{1988} - 2^{-1988}$, compute $P^2 - Q^2$.

F88J4. Each side of a regular hexagon is twelve units long. A circle is drawn circumscribing the hexagon, and another inscribed in the hexagon. Compute the area of the region between the two circles.

Part III: 10 Minutes NYCIML CONTEST ONE FALL, 1988

F88J5. Find all ordered triples (a, b, c) of real numbers such that $a + b = 7$, $b + c = 12$ and $a + c = 3$.

F88J6. Goldbach's conjecture, not yet proven, states that every even number can be represented as the sum of two prime numbers. Find the smallest possible <u>difference</u> between two prime numbers whose sum is 98.

ANSWERS

1. 1/27
2. 3
3. 4.
4. 36π
5. (-1, 8, 4): ordered triple required
6. 24

New York City Interscholastic Mathematics League

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE

JUNIOR DIVISION — CONTEST NUMBER TWO — FALL, 1988

Part I: 10 Minutes

F88J7. One-quarter is the same part of one-third as one-half is of what number?

F88J8. Two ants are racing around the perimeter of a unit square. They start at the same vertex and both go clockwise. One ant runs at the constant rate of one unit per second, while the second travels at the constant rate of two units per second. After 17 seconds, how far apart will they be?

Part II: 10 Minutes — NYCIML CONTEST TWO — FALL, 1988

F88J9. Find all ordered triples (a, b, c) of <u>positive</u> real numbers such that $\sqrt{ab} = 5$, $\sqrt{ac} = 8$, and $\sqrt{bc} = 2$.

F88J10. In triangle ABC, AB = BC. Point D is on AB and CD bisects angle ACB. The perpendicular to CD at D intersects line CA at E. If AD = 1, find the length of EC.

Part III: 10 Minutes — NYCIML CONTEST TWO — FALL, 1988

F88J11. A square kilometer of land in New Jersey has become contaminated with radioactive waste. Because of the radiation, no one can live within one kilometer of any point within the contaminated area. Find the number of square kilometers in the uninhabitable area (including the interior of the square).

F88J12. A three digit number is reversed (so that the units digit and the hundreds digit are interchanged), to form a new and <u>larger</u> number. The product of the original smaller number and the new larger number is 65125. Find the original smaller number.

ANSWERS

7. 2/3
8. 1
9. (20, 5/4, 16/5)
10. 2
11. $5 + \pi$
12. 125.

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE
JUNIOR DIVISION CONTEST NUMBER THREE FALL, 1988

Part I: 10 Minutes

F88J13. In 1988, the population of Abra increased by 20%, while the population of Cadabra decreased by 10%, after which the two populations were equal. What percentage of the original population of Cadabra was the original population of Abra?

F88J14. If a = 10000000000, compute $\sqrt[13]{\left(a\sqrt[5]{a^8}\right)}$.

Part II: 10 Minutes NYCIML CONTEST THREE FALL, 1988

F88J15. Express as a finite or repeating decimal the number whose binary representation is 1.01010101..., where the dots indicate infinite repetition.

F88J16. Square ABCD is drawn on a rectangular coordinate plane. Its vertices are labelled in clockwise order. If the coordinates of A are (2,0) and the coordinates of B are (0,1), find the ordered pair (x,y) of coordinates of point C.

Part III: 10 Minutes NYCIML CONTEST THREE FALL, 1988

F88J17. A solid cube of radium is floating in deep space. Each edge of the cube is exactly one kilometer in length. An astronaut is protected from its radiation if she wears a velcro suit and remains at least one kilometer from the nearest speck of radium. Including the interior of the cube, what is the volume (in cubic kilometers) of space which is forbidden to such an astronaut?

F88J18. The numbers a, b, c, d are such that the sum of any one of then and the product of the other three is equal to 2. If a > b, a > c and a > d, find a.

ANSWERS

13. 75 or 75%.

14. 100

15. 1.333... or equivalent

16. (1,3)

17. $7 + 13\pi/3$

18. 3.

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE
JUNIOR DIVISION CONTEST NUMBER ONE FALL, 1988

SOLUTIONS

F88J1. Since there are 3 feet in one yard, there are 3x3x3 = 27 cubic feet in one cubic yard.

F88J2. Since $72 = 9 \cdot 8$, the given number must be a multiple of 9 and also of 8. To get a multiple of 9, we must have $1+2+8+7+x+y+6$ a multiple of 9, so $x+y$ must have a remainder of 3 upon division by 9. To get a multiple of 8, the number represented by xy6 must be a multiple of 8, so the number y6 must be a multiple of 4. This requires that y be 1, 3, 5, 7 or 9. For $y = 9$, we have $x = 3$, and 396 is not a multiple of 8. For $y = 5$, we have $x = 7$, and 756 is not a multiple of 8. For $y = 7$, we have $x = 5$, for $y = 3$, we have $x = 0,9$, for $y = 1$, we find $x = 2$, and of these only (5,7), (9,3) and (2,1) work.

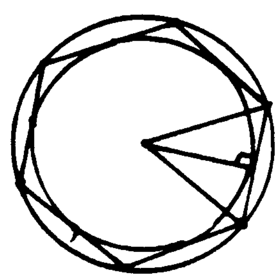
F88J3. Let $2^{1988} = x$, $2^{-1988} = y$, so that $xy = 1$. Then $P^2 = x^2 + 2xy + y^2$, while $Q^2 = x^2 - 2xy + y^2$, and $P^2 - Q^2 = 4xy = 4$.

F88J4. The radius of the larger circle is 12. The radius of the smaller circle is an altitude of one of the equilateral triangles forming the hexagon (see diagram), which is $6\sqrt{3}$.

The required area is $\pi(R^2 - r^2) = \pi(144 - 108) = 36\pi$.

F88J5. Adding the three equations shows that $2(a+b+c) = 22$, so $a+b+c = 11$. Subtracting each of the given equations from this derived relationship gives the required answer.

F88J6. Since $98 = 2 \cdot 49$, we seek a prime number p as close as possible to 49, such that the difference $98 - p$ is also prime. Trying in succession 47, 43, 41, we find that $98 - p$ is not prime. But for $p = 37$, $98 - p = 61$. The difference is 24.



NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE
JUNIOR DIVISION CONTEST NUMBER TWO FALL, 1988

SOLUTIONS

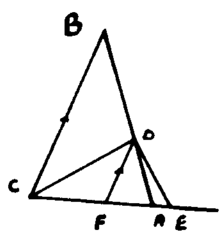
F88J7. $(1/4)/(1/3) = (1/2)/x$, or $3/4 = 1/(2x)$, or $6x = 4$, and $x = 2/3$.

F88J8. Label the vertices A, B, C, and D, in clochwise order, and suppose the ants start at vertex A. Then after 17 seconds the first ant will be at vertex B, and the second ant will be at vertex C. They will be one unit apart.

F88J9. Compare problem F88J5. Here, to preserve the symmetry of the given equations, we can multiply them together. We find that $\sqrt{a^2b^2c^2} = 80 = |abc|$, so that abc = 80 (since all are positive). Squaring each of the given equations, and dividing this new relationship by the squared equations gives the result.

F88J10. Draw DF || BC (see diagram). This construction forms three isosceles triangles:

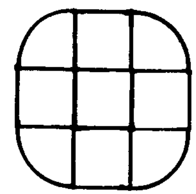
First, since triangles ADF, ABC are similar, triangle ADF is isosceles, so that AD = DF. Also, <DCF = <BCD = <CDF, so triangle CFD is isosceles, with CF = FD. Finally, <FDE = 90 - <CDF = 90 - <DCE = = <DEF (in right triangle CDE). Hence triangle DEF is isosceles, with FD = FE.



Therefore, AD = DF = FE = CF = 1, and CE = 2.

Note that if angle B is obtuse, the diagram looks somewhat different.

F88J11. The required area includes five squares of side 1 (see diagram) and four quadrants of a circle with radius 1. These form a complete circle, whose area is $\pi 1^2 = \pi$. The total area is thus $5 + \pi$.



F88J12. Since the units digit of the product is 5, the units digit of the original number must be 5. Hence the hundreds digit of the new number is also 5.

Since 500 x 200 = 100 000, and since the product doesn't have this many digits, the hundreds digit of the original number must be 1.

```
  1b5
  5b1
  ---
  1b5
    c
-----
65125
```

Finally, examining the multiplication algorithm for the product in question (see diagram), we find that the digit called c must be 0 or 5 (since it is the last digit of a multiple of 5). If c = 5, then b = 7, which does not check. But if c = 0, b = 2, and 125x521 = 65125.

SOLUTIONS

F88J13. If A and C represent the original populations of Abra and Cadabra respectively, we have 1.2A = .9C, so A/C = 9/12 = 3/4, and A = .75C.

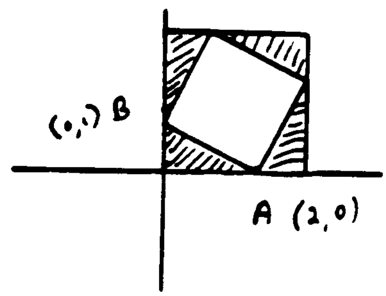
F88J14. We have $\sqrt[13]{\left(a\sqrt[5]{a^8}\right)} = \sqrt[13]{\left(\sqrt[5]{a^{13}}\right)} = a^{((13/5)(1/13))} = a^{1/5}$. Since $a = 10^{10}$, the answer to the problem is 10^2, or 100.

F88J15. If B = 1.01010101..., then (writing the coefficient in decimal notation),

4B = 101.01010101..., and subtracting we find that
3B = 100 (base 2) = 4 (decimal).

Hence B = 4/3 = 1.333... or $1.\overline{3}$, or equivalent.

F88J16. We can circumscribe the required square with another square, whose sides are parallel to the coordinate axes. Then it is clear the the four shaded triangles (see diagram) are congruent, and we can read off the coordinates of point C immediately.



F88J17. Compare problem S88J11. The required volume breaks down into seven cubes of unit volume, twelve quarters of a right circular cylinder with unit radius and unit height, and eight octants of a sphere with unit radius. The total volume is thus $7 + 12(\pi/4) + 4\pi/3 = 7 + 3\pi + 4\pi/3 = 7 + 13\pi/3$.

F88J18. We have:

$$a + bcd = 2$$
$$b + acd = 2$$
$$c + abc = 2$$
$$d + abc = 2.$$

Subtracting the first two equations, we find $(a-b) + (b-a)cd = 0$, or $(a-b)(1-cd) = 0$. Since $a - b \neq 0$, we must have $cd = 1$. Treating other pairs of equations in the same way (or, noting the symmetry in b, c, and d), we find that $bc = 1$ and $bd = 1$. We can now solve the three symmetric equations $bc = 1$, $cd = 1$, $bd = 1$ (compare problem F88J9). Multiplying them together, we find $bcd = 1$ or -1. Taking $bcd = 1$ leads to $b = c = d = 1$, and $a = 1$ as well. This contradicts the conditions of the problem. Hence we must take $bcd = -1$, which leads to $b = c = d = -1$, and $a = 3$.