Spring 1988 – Senior B – 1 – Questions

S88B1

Find all possible non-negative remainders when the square of an even number is divided by 8.

S88B2

If $x = \left(1 + i\sqrt{5}\right)/2$, where $i^2 = -1$, compute the numerical value of $x^2 - x$.

S88B3

Find all real numbers *x* such that $x^2|x| = 8$.

S88B4

Two adjacent sides of a parallelogram are 3 inches and 4 inches in length, and the area of the parallelogram is $3\sqrt{7}$ square inches. Find the inch-length of the longer diagonal of the parallelogram.

S88B5

In trapezoid ABCD, base AB = 9 cm., base CD = 15 cm., and the altitude is 4 cm. Legs AD and BC are extended to meet at point E. If F is the midpoint of AD and G is the midpoint of BC, find the number of square centimeters in the area of triangle FGE.

S88B6

Find all real numbers *x* such that $8^{2x^2-7x+3} = 4^{x^2-x-6}$.

Answers

1. 0, 4: both required
2. –3/2
3. 2, -2: both required
4. $\sqrt{43}$
5. 48 or 48 cm^2
6. 7/4, 3: both required

Spring 1988 – Senior B – 2 – Questions

S88B7

If 7/5 is expressed as $a+\frac{1}{b+\frac{1}{c}}$, where *a*, *b*, and *c* are positive integers, find *c*.

S88B8

An arithmetic sequence contains n terms, where n > 2, and all of the terms of the sequence are inters. If the first term is 25, and the last term is 31, find all possible values of n.

S88B9

How many ordered triples (x,y,z) of positive integers are there such that x < y < z, and xyz = 105?

S88B10

Three points A, B, and C, are non-collinear. Circles are drawn with AB and AC as diameters. These circles intersect at points A and D. If the inch-lengths of BD and CD are 8 and 7 respectively, find the inch-length of BC.

S88B11

The diagonals of a rhombus are 10 and 24 inches long. Find the inch-length of a radius of the circle inscribed in the rhombus.

S88B12

Find all ordered pairs (x,y) of positive integers such that $x^2 - y^2 = 104$.

Answers

7. 2
8. 3, 4, 7: all three required
9. 4
10. 15
11. 60/13 or equivalent
12. (27,25); (15,11) both ordered pairs required

Spring 1988 – Senior B – 3 – Questions

S88B13

If a:b:c = 2:3:4, compute (a/b):(b/c).

S88B14

In an arithmetic sequence of n terms are integers, the common difference is an integer. If the 3^{rd} term of the sequence is 4, and the $(n-2)^{nd}$ term is 9, find the n^{th} term.

S88B15

If $a = \log_8 49$, and $b = \log_2 7$, compute the ratio a:b.

S88B16

For <u>how many</u> pairs (x,y) of positive integers is $x^2 - y^2 = 156$?

S88B17

If $a^2 + a + 1 = 0$, compute the numerical value of $a^4 + a^2$.

S88B18

How many non-congruent triangles are there with sides which are positive integer, and whose perimeter is 15?

Answers

13. 8/9
14. 11, 19, or -1
15. 2/3 or 2:3 or equivalent
16. 3
17. –1
18. 7

Spring 1988 – Senior B – 4 – Questions

S88B19

The length of a side of an equilateral triangle is 6. A circle is inscribed in the triangle, and a square is inscribed in the circle. Compute the length of a side of the square.

S88B20

If, for all real number x, $f(x) = 8x - 3$, find the ordered pair (c,d) of real numbers such that $f(c) = d$ and $f(d) = 10c$.

S88B21

The sequence 3, a, b is a geometric progression. The sequence a, b, 9 is an arithmetic progression. Compute the positive real value of a.

S88B22

Find the sum of all positive integers less than 100 which are divisible by 3, but not divisible by 2.

S88B23

In circle O, an arc of $60°$ has the same length as an arc of $45°$ in circle P. Compute the ratio of the radius of circle O to the radius of circle P.

S88B24

Find all real numbers x for which there exists some positive number k which is not equal to 1, and for which $(\log_k x)(\log_5 k) = 5/2$.

Answers

19. $\sqrt{6}$
20. (1/2, 1) ordered pair required
21. 9/2
22. 867
23. 3/4
24. $25\sqrt{5}$ or equivalent

Spring 1988 – Senior B – 5 – Questions

S88B25

Compute $33(3+3\times 3-3\div 3+3)$.

S88B26

An infinite geometric progression with common ratio *r* (where $|r|<1$) has an (infinite) sum of 15. The series whose terms are the squares of the terms of the original sequence has an (infinite) sum of 45. Compute the numerical value of *r*.

S88B27

If x:y = 4:3, express in lowest terms the ratio $(x^2 - y^2):(x^2 + y^2)$.

S88B28

A student made a list of the positive prime numbers under 1000, in increasing order. She then multiplied two consecutive elements of her list, and got 34571. Find the smaller of the two numbers the student multiplied together.

S88B29

If $2(4^x)+6^x = 9^x$ and $x = \log_{2/3} a$, find the numerical value of a.

S88B30

What is the minimum value of the sum of the squares of the distances of a point to the vertices of a triangle whose sides are 3, 4, and 5?

Answers

25.	462
26.	2/3
27.	7/25
28.	181
29.	1/2
30.	50/3

Spring 1988 – Senior B – 1 – Solutions

S88B1

An even number can be written as either 8n, 8n + 2, 8n + 4, or 8n + 6. Squaring each of these expressions, we find that each is either a multiple of 8 or four more than a multiple of 8.
This result is easily conjectured by trying a few even squares.

S88B2

By direct computation, we find that $x^2 = -1 + (1/2)i\sqrt{5}$. Thus $x^2 - x = -1 - 1/2 + i\sqrt{5}/2 - i\sqrt{5}/2 = -3/2$.

S88B3

If $x > 0$, we have $x^3 = 8$, and $x = 2$. If $x < 0$, we have $x^3 = 8$, and $x = -2$.

S88B4

The altitude to the side of length 3 has inch-length $\sqrt{7}$. Hence in right triangle BXC (see diagram), the Pythagorean theorem shows that CX = 3. Then the same theorem used in right triangle BXD shows that BD is $\sqrt{43}$.

S88B5

Note that FG is the average of AB and CD, which is l2. If EX = x (see diagram), then, from similar triangles EAB, EDC, x:9 = (x + 4):15, or 9x + 36 = 15x, which gives x = 6. Then, since FG bisects any altitude of the trapezoid, EZ = x + 2 = 8, and the area of triangle EFG is $(1/2)\mathrm{EZ} \cdot \mathrm{FG} = 4 \cdot 12 = 48$.

[diagram goes here]

S88B6

Since $8 = 2^3$, we can rewrite the equation as $2^{6x^2-21x+9} = 2^{2x^2-2x-12}$. Equating the exponents leads to $4^2 - 19x + 21 = (4x - 7)(x - 3) = 0$, and $x = 7/4$, 3.

Spring 1988 – Senior B – 2 – Solutions

S88B7

$7/5 = 1 + 2/5 = 1 + \dfrac{1}{5/2} = 1 + \dfrac{1}{2 + \frac{1}{2}}$, so c = 2.

S88B8

Let the common difference of the sequence be d. Then 31 = 25 + (n – 1)d, or d(n – 1) = 6. Since both factors on the left are integers, we can try pairs of factors of 6. We find only d = 1, n = 7; or d = 2, n = 4; or d = 3, n = 3 (if d = 6, n = 2, which contradicts the conditions of the problem). This gives three sequences:
25, 26, 27, 28, 29, 30, 31
25, 27, 29, 31
25, 28, 31.

S88B9

Since $105 = 3 \cdot 5 \cdot 7$, we must have (x,y,z) = (3,5,7), or else x = 1, and yz = 105. The latter case leads to the ordered triples (1,3,35), (1,5,21), (1,7,15),for four ordered triples in all..

S88B10

Drawing AD (see diagram), we find that angles ADC and ADB are both right angles (since each is inscribed in a semicircle). Hence angle BDC is a straight angle, and points B, D, and C are collinear. Hence BC = BD + DC = 8 + 7 = 15

S88B11

The problem reduces to finding the altitude to the hypotenuse of a 5-12-13 right triangle (see diagram). Computing the area in two different ways, we find 13x/2 = $5 \cdot 12/2$, and x = 60/13.

S88B12

We have (x + y)(x – y) = 104. Since both factors on the left are integers, we can solve this by setting these factors equal to all possible pairs of factors of 104. Since x + y > x – y, we need only consider pairs of factors in which the first factor is the larger. Also, since x is the average of (x + y) and (x – y), and x must turn out an integer, we need only consider pairs of factors which are either both even (they cannot be both odd, as 104 is even). This leads to the following table:

104	x + y	x – y	2x	x	y
$52 \cdot 2$	52	2	54	27	25
$26 \cdot 4$	26	4	30	15	11

Spring 1988 – Senior B – 3 – Solutions

S88B13

Let a = 2x, b = 3x, c = 4x. Then

$(a / b):(b / c) = (a / b)(c / b) = ac / b^2 = (2x)(4x)/(3x)^2 = 8/9$.

S88B14

See problem S87B8. If the first term of the sequence is u, and the common difference is d, then u + 2d = 4, u + (n – 3)d = 9, and (subtracting the two equations), d(n – 3 – 2) = 5, or d(n – 5) = 5. Since both terms on the left are positive integers, we can only have d = 1, n = 10, and the last term is u + d = 11.

S88B15

We have $8^a = 49 = (2^3)^a = 2^{3a}$, so $2^{3a/2} = 7$, and $\log_2 7 = 3a/2$. Thus a/b = 2/3.

S88B16

See problem S87B12. There are as many ordered pairs (x,y) as there are pairs of factors of 256, both of which are even, and the first of which is not smaller than the second. Since $256 = 2^8$, the only such factorizations are those in which both factors are positive integral powers of 2: $2 \cdot 2^7$, $2^2 \cdot 2^6$, $2^3 \cdot 2^5$, $2^4 \cdot 2^4$. This gives four ordered pairs.

S88B17

We have: $a^2 + a + 1 = 0$. Multiplying by a, $a^3 + a^2 + a = 0$. Subtracting, $a^3 - 1 = 0$, so $a^3 = 1$. Thus $a^4 + a^2 = a^2(a^2 + 1)$. But the original equation are the complex cube roots of 1. Note that there are two of them, and that it does not matter, for this problem and almost any other problem, which of the two we are thinking of.

S88B18

If the length of the sides of such a triangle are *a*, *b* and *c*, we can assume that $c \le b \le a$. We must find those ordered triples which satisfy the triangle inequality, or $a < b + c$. Noting that *c* cannot be greater than 5, we can examine these different cases:

c = 1: Then $a + b = 14$. If *b* = 7, *a* = 7.
If *b* = 6, *a* = 8 and *b* + *c* < *a*.
If *b* < 6, the situation gets worse.

c = 2: Then $a + b = 13$. If $b = 6$, $a = 7$.
If $b = 5$, $a = 8$ and $b + c < a$.
If $b < 5$, the situation gets worse.

c = 3: Then $a + b = 12$. If *b* = 6, *a* = 6.
If *b* = 5, *a* = 7
If *b* = 4, *a* = 8 and *b* + *c* < *a*.
If *b* < 4, the situation gets worse.

c = 4: Then $a + b = 11$. If *b* = 5, *a* = 6.

If $b = 4$, $a = 7$
If $b = 3$, $a = 8$ and $b + c < a$.
If $b < 3$, the situation gets worse.
$c = 5$: Then $a + b = 10$. If $b = 5$, $a = 5$.
If $b < 5$, then $b < c$.

There are seven solutions in all: (1,7,7), (2,6,7), (3,6,6), (3,5,7), (4,5,6), (4,4,7), (5,5,5).

Spring 1988 – Senior B – 4 – Solutions

S88B19

In the diagram, we first see a 30-60-90 right triangle, which shows that a radius of the circle is $\sqrt{3}$. The second diagram shows that a side of the square is $\left(\sqrt{3}\right)\left(\sqrt{2}\right) = \sqrt{6}$.

[diagram goes here]

S88B20

We have $\begin{matrix} 8c - 3 = d \\ 8d - 3 = 10c \end{matrix}$. Solving simultaneously, we find $c = 1/2$ and $d = 1$.

S88B21

We have $a^2 = 3b$, and $2b = 9 + a$. Thus $2a^2 = 6b = 27 + 3a$, so $2a^2 - 3a - 27 = 0$. This quadratic has roots 9/2 and –3. The second solution is extraneous.

S88B22

The multiple of 3 within the given range form an arithmetic sequence whose first term is 3, last term is 99, and whose common difference is 3. It is not hard to see that there are 33 terms in all, so their sum is (33/2)(99 + 3) = $33 \cdot 51$.
From this sum, we must subtract the sum of those integers which are multiples of both 3 and 2. These are exactly the multiples of 6, and they again form an arithmetic sequence, with first term 6. Last term 96, and common difference 6. There are 16 of them, and their sum is (16/2)(96 + 6) = $16 \cdot 51$.
The desired sum is thus $33 \cdot 51 - 16 \cdot 51 = 17 \cdot 51 = 867$.

S88B23

If the radius of circle O is *r*, and the radius of circle P is *s*, then $\left(1/6\right)\left(2\pi r\right) = \left(1/8\right)\left(2\pi s\right)$, or $r/6 = s/8$, so $r/s = 3/4$.

S88B24

Since $\left(\log_a b\right)\left(\log_b c\right) = \log_a c$, the left side of the given equation reduces to $\log_5 x = 5/2$ and $x = 5^{5/2} = 25\sqrt{5}$.

Spring 1988 – Senior B – 5 – Solutions

S88B25

The given expression equals $33(3+9-1+3) = 33(14) = 462$.

S88B26

The squares of the terms of a geometric progression form another geometric progression. Therefore, if the first term of the original sequence is a, the $a/(1-r) = 15$, and $a^2/(1-r^2) = 45$. Dividing these two equations, we find that $a/(1+r) = 3$. Multiplying this by the first equation, we find that $(1+r)/(1-r) = 5$, which leads to $r = 2/3$.

S88B27

Since $y = 3x/4$, $\begin{matrix} x^2 - y^2 = x^2 - (9x^2)/16 \\ x^2 + y^2 = x^2 + (9x^2)/16 \end{matrix}$, and the ratio of these two quantities is (1 – 9/16):(1 + 9/16) = (16 – 9):(16 + 9) = 7:25.

S88B28

The two prime factors of the number 34571 cannot be too far apart, since they are consecutive primes less than 1000. In fact, they cannot be too far from the square root of this number. Let us first find the tens and hundreds digits of the required numbers. We find that $180^2 = 32400$, $190^2 = 36300$, so that the numbers are likely to be around 185. Since they are prime, and their product ends in the units digit 1, they must have units digits of 1 and 1, or of 3 and 7. But 183 a multiple of 3, while 193 is a multiple of 7, so it is best to try 181 and 191. In fact, these are both prime, and $181 \cdot 191 = 34571$.

S88B29

$2^x \cdot 4^x + 6^x = 9^x$, so $2^{2x+1} + 2^x \cdot 3^x - 3^2 = 0$, or $(2^{x+1} - 3^x)(2^x - 3^x) = 0$, so $2^{x+1} = 3^x$ (since $2^x = -3^x$ is impossible). Thus $(2/3)^x = 1/2$, so $x = \log_{2/3} 1/2$, and $a = 1/2$.

S88B30

Suppose the vertices of the triangle are at (0,0), (4,0) and (0,3). If (x,y) is the point where the minimum occurs, we must have a minimal value for $x^2 + y^2 + (x-4)^2 + y^2 + x^2 + (y-3)^2 = 3x^2 - 8x + 16 + 3y^2 - 6y + 9$. We can minimize each quadratic separately, hence their sum, by taking $x = 4/3$, $y = 1$ (think of the formula for the turning point of a parabola). This gives 50/3 for the minimal value.

By generalizing the above argument, it can be shown that the minimal value will always occur at the centroid (intersection of the medians) of the right triangle, no matter what kind of triangle is chosen.