Fall 1987 – Senior A – 1 – Questions

F87S1

Find the largest four-digit number (in decimal notation), which is a multiple of 3, and whose digits are all even (but not necessarily distinct).

F87S2

If $f(x) = \sqrt{x-7}$, find all real numbers x such that $f(f(x)) = 3$.

F87S3

A small city has a single subway line, with 15 stations. To travel from one station to another on this line, a passenger must purchase token, whose weight is proportional to the length of the intended trip. If no two pairs of stations on the line are the same distance apart, how many kinds of tokens must be sold?

F87S4

In a rectangular coordinate system, the vertices of triangle ABC are at A(3,3), B(7,7), and C(12,3). How many points are on the perimeter of this triangle whose coordinates are both integers?

F87S5

If A = 1155, how many positive integers B are there such that $A + B^2$ is a perfect square?

F87S6

In a parallelogram ABCD, $\sin \angle ADC = 1/3$, AD = 9, and DC = 12. The four bisectors of the interior angles of ABCD enclose a quadrilateral. Find the area of this quadrilateral.

Answers

1. 8886
2. 263
3. 105
4. 14
5. 8
6. 3/2

Fall 1987 – Senior A – 2 – Questions

F87S7

In the land of Malfortuna, the residents go on a picnic every third day, it rains every fourth day, and there is a palace coup every seventh day. If all three events occurred on January 1 of a certain non-leap year, on what day will they next all occur again on the same day?

F87S8

Points A, B, and C lie on a circle with radius 2, and $m\overset{\frown}{AB} : m\overset{\frown}{BC} : m\overset{\frown}{AC} = 3:4:5$. Tangents to the circle are drawn, with A, B, C as points of contact. These tangents form a triangle. Find the area of this triangle.

F87S9

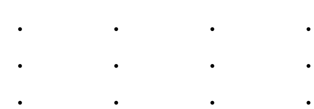
Rectangle KLMN is inscribed in triangle ABC, so that K and L are on side AB, M is on side BC, and N is on side AC. If CD is an altitude of triangle ABC, compute the numerical value of the sum MN:AB + KN:CD.

F87S10

John took a positive integer, cubed it, and passed it to Bob. Bob divided the integer by 7, forgot the remainder, and passed the quotient back to John. John divided this number by 7, forgot the remainder, and passed the quotient back to Bob. Bob divided by 7 again, forgot the remainder, and passed the quotient back to John. This last quotient was 9. What was John's original number?

F87S11

Pictured below is a rectangular array of 12 dots, of which 10, or 5/6 of the total number, are on the perimeter. Find the largest n such that a rectangle of n dots can be formed, with half of the dots on the perimeter.



F87S12

Find the maximum value of $3\sin 2x + 4\cos 2x$, for all real numbers x.

Answers

7. March 26 or 3/26 r 26/3 (or equivalent <u>date</u>)
8. $12 + 8\sqrt{3}$
9. 1
10. 15
11. 60
12. 5

Fall 1987 – Senior A – 3 – Questions

F87S13

In 1932, Franklin noticed that his age was the same as the last (rightmost) two digits of the year of his birth. His grandfather Teddy noticed the same thing about his own age. Franklin was born in the twentieth century. How old was Teddy when Franklin was born?

F87S14

If the argument of the sine function is written using radian measure, then which of the following numbers is largest: sin1, sin2, sin, or sin47.

F87S15

In triangle ABC, points X and Y are on AC (with X closer than Y to point A) and points P and Q are on AB (with P closer than Q to point A). The broken path AP x QY x BC divides the triangle into five small triangles of equal area. If AC = 30, find the length of XY.

F87S16

If x + (1/x) = a, express as a polynomial in a the value of the expression $x^5 + \left(1/x^5\right)$.

F87S17

If $\log_4\left(\log_3\left(\log_2 x\right)\right) = 1$, the real number x can be expressed as 2^k, where k is a positive real number. Compute the value of k.

F87S18

In triangle ABC, AB = 7, AC = 8, and BC = 5. Point M is chosen on side BC, and perpendiculars MN, MP are drawn to sides AB and AC respectively. For all such choices of M, M_o is that point for which PN is shortest. Find the ratio $BM_o : M_oC$.

Answers

13. 50
14. sin2
15. 8
16. $a^5 - 5a^3 + 5a$ or equivalent polynomial
17. 81
18. 1:4 or equivalent

Fall 1987 – Senior A – 4 – Questions

F87S19

The product of 1000 natural numbers (not all distinct) is equal to 1000. What is the maximum possible sum of these 1000 numbers?

F87S20

In rhombus ABCD, side AB is the geometric mean (mean proportion) between diagonals AC and BD. Find e degree-measure of the smallest interior angle of the rhombus.

F87S21

Frivolous Fred bought a plot of land in the desert, which was supposed to be exactly rectangular in shape. The plot was marked by four poles, at the vertices of the supposed rectangle. He hired Ajax Surveyors to find out whether the plot was indeed rectangular. The surveyors do not know how to measure angles, and they cannot determine points of intersection of lines. In fact, all they can do is to measure the distance between two points. They charge $1 to measure the distance between one pair of points. At most, how many dollars will Freddy have to spend to find out whether or not his plot is rectangular?

F87S22

In triangle ABC, AB = c, AC = b, and BC = a. If $c^2 = \left(a^3+b^3+c^3\right)/\left(a+b+c\right)$, find the degree-measure of angle C.

F87S23

Find all real numbers or complex numbers x which satisfy (x – 1)(x – 3)(x + 5)(x + 7) = 297.

F87S24

A circle of radius 3 inches is tangent internally at points P to a circle of radius 6 inches. The smaller circle rolls without slipping around the inside of the larger circle, until its new point of contact is diametrically opposite the original point P. An ant rides on the smaller circle sitting on the point which was originally at P. How long a ride, in inches, will the ant take?

Answers

19. 1999
20. 30 or $30°$
21. 5 or $5
22. 60 or $60°$
23. –8, 4, $-2 \pm i\sqrt{2}$
24. 12 or 12 inches

Fall 1987 – Senior A – 5 – Questions

F87S25

The product of two or more natural numbers, none of which is equal to 1, is 24. What is the largest sum that these natural numbers could have?

F87S26

In triangle ABC, AB < BC < CA, and the lengths of the three sides are consecutive integers each greater than 3. The altitude to BC divides that side into two segments. Find the (positive) difference between the lengths of these two segments.

F87S27

If f(x) is a function defined on the real numbers such that f(2) = 5, and for all real x, f(x)f(x + 1) = 3, find f(10).

F87S28

In square ABCD, side AB = 4, P is the midpoint of side AB, and Q is the point on DC such that DQ:QC = 3:1. AQ and PD intersect in X, and CP and BQ intersect in Y. Find the area of quadrilateral PXQY.

F87S29

Find all real or complex numbers x such that:

$$\frac{1}{x^2+2x-3}+\frac{18}{x^2+2x+2}=\frac{18}{x^2+2x+1}$$

F87S30

A right circular cylinder has a radius of 5 and a height of 3. Four non-coplanar points are chosen on the lateral surface of the cylinder. The points may be on the “rim” of the cylinder, but not in the interior of a base. Find the largest possible volume of the tetrahedron formed by these four points.

Answers

25. 14
26. 4
27. 5
28. 56/15
29. 2, -4, $-1\pm 2\sqrt{2}$: all four required
30. 50

Fall 1987 – Senior A – 1 – Solutions

F87S1

For the number to be divisible by 3, the sum of its four digits must be a multiple of 3. To make the number as large as possible, it should have 8's in the highest places. But 8888 is not a multiple of 3: in fact, it has a remainder of 2 when divided by 3. The next lowest multiple of 3 is thus 8886.

F87S2

We have $\sqrt{\sqrt{x-7}-7}=3$, or $\sqrt{x-7}-7=9$, so $\sqrt{x-7}=16$. Then $x-7=256$, so $x=263$.

F87S3

From any one station, there are fourteen other stations which can be reached. Hence, each station must carry fourteen different types of tokens. This would result in 15 x 14 = 210 different types of tokens. However, the same token can be used for the trip between any two stations and the return trip, so there are really only half as many tokens required, or 105.

F87S4

A point whose coordinates are both integers is called a lattice point.
Line segment AC lies along line y = 3, so that its lattice points are easily counted: there are 10 of them.
Line segment AB lies along e line y = x, so its lattice points are again easily counted: there are four of them, not counting vertex A (which was already counted above).
Line segment BC lies along a line whose slope is –4/5. Hence if we start at point B, and move four units down and five to the right, we will arrive at another lattice point. Since the fraction –4/5 is in lowest terms, we will not be skipping any lattice points. Since one such move brings us from vertex B (which was already counted) to vertex C (which was already counted), we conclude that there are no more lattice points on this side.
Hence there are 14 lattice points altogether.

F87S5

Suppose $S^2 = A + B^2$. Then $A = B^2 - S^2 = (B+S)(B-S) = 1155$. Since A factors into primes as 3 x 5 x 7 x 11, we can account for all solutions by matching this factored expression with all possible numerical factorizations of A. Since B + S > B – S (we can assume that S is positive), and since $1155 < 1156 = 34^2$, we can assume that B – S < 34. This leaves the following possibilities:

B – S	B + S	B
3	385	194
5	231	118
7	165	86
11	105	58
15	77	46

21	55	38
33	35	34
1	1155	578

There are eight such possibilities.

F87S6

In general, let AD = b, CD = a, $\angle ADC = x$. It is not hard to see that the quadrilateral described is in fact a rectangle, so the required area is PQ x PS (see diagram). Let us compute these lengths.

In right triangle ABS, $A = AB\sin\angle ABS = a\sin x/2$.

In right triangle APD, $AP = AD\sin\angle ADP = b\sin x/2$, so $PS = (a\sin x/2 - b\sin x/2)$.

In an analogous fashion, we find that $PQ = (a\cos x/2 - b\cos x/2)$.

Hence the product

$$PQ\cdot PS = (a\cos x/2 - b\cos x/2)(a\sin x/2 - b\sin x/2)$$
$$= a^2\cos x/2\sin x/2 - 2ab\cos x/2\sin x/2 + b^2\cos x/2\sin x/2 .$$

But $\cos x/2\sin x/2 = (1/2)\sin 2x/2 = (1/2)\sin x$. Factoring this out of the above expression, we find that $PS\cdot PQ = (1/2)\sin x\left(a^2 - 2ab + b^2\right) = (1/2)(a-b)^2\sin x$.

In the present case, this equals $(1/2)(9)\cdot(1/3) = 3/2$.

Must rectangle PQRS be contained within the original parallelogram?

Fall 1987 – Senior A – 2 – Solutions

F87S7

The number of days which must elapse between coincidences of the three events is the least common multiple of the three numbers. Since 3, 4, and 7 have no common divisor (other than one), this is simply their product, which is 84. The required day is thus the 85th day of the year. Since January has 31 days, and February has 28 (in a non-leap year), these two months account for 59 days, and we need 26 more days from the month of March.

F87S8

It follows that $m\widehat{AB} = 90$, $m\widehat{BC} = 120$, and $m\widehat{AC} = 150$. If the required triangle is XYZ, then, since the angle formed by two tangents is the supplement of its minor arc, $m\angle Z = 90$, $m\angle Y = 60$ and $m\angle X = 30$. Then, drawing OY (see diagram), and using 30-60-90 triangle BOY, we have BO = BZ = 2, so $BY = 2\sqrt{3}$, and $YZ = 2 + 2\sqrt{3}$. Then, from 30-60-90 triangle XYZ, $XZ = 6 + 2\sqrt{3}$. Finally, the required area is $(1/2)ZX \cdot ZY = (1/2)(2+2\sqrt{3})(6+2\sqrt{3}) = 12 + 8\sqrt{3}$.

F87S9

From similar triangles ABC, NMC, NM:AB = NC:AC. From similar triangles KNA, DCA, NK:CD = AN:AC. Adding, we find that NM:AB + NK:CD = NC:CA + AN:AC = AC:AC =1.

F87S10

We can trace this process backwards.
If John got 9, Bob must have passed him at least 63, and at most 69.
If Bob got 63, John must have passed him at least 441.
If Bob got 69, John must have passed him at least 489.
If John got 441, Bob must have started with at least 3087.
If John got 489, Bob must have started with at least 3429.
Since $14^3 = 2744$, $15^3 = 3375$, and $16^3 = 4096$, John must have begun by cubing 15.

F87S11

If there are x rows of y dots, then xy = n, and 2x + 2(y – 2) = n/2 = xy/2. This gives x = 4 + 8/(y – 4).
Since x is positive, y > 4. Hence y – 4 must divide 8, so y – 4 = 1, 2, 4, or 8, and y = 5, 6, 8, and 12. This gives rectangles with 48 or 60 dots, and the largest n is 60.

F87S12

If $f(x) = 3\sin 2x + 4\cos 2x$, then $(1/5)f(x) = (3/5)\sin 2x + (4/5)\cos 2x$. Since $(3/5)^2 + (4/5)^2 = 1$, we can find a number A such that $\cos A = 3/5$ and $\sin A = 4/5$.

Then $\left(1/5\right)f\left(x\right) = \cos A \sin 2x + \sin A \cos 2x = \sin\left(2x + A\right)$, and $\left|\left(1/5\right)f\left(x\right)\right| \le 1$, with equality when $2x + A = \pi/2$, or $\pi/4 - A/2$.

Hence the maximum value of $f\left(x\right)$ is 5.

Fall 1987 – Senior A – 3 – Solutions

F87S13

The year of Franklin's birth contains four digits, of which the first two are 19. The last two are also his age, so added to themselves, they must give 32. Hence Franklin was born in 1926, and is 16 years old.
Clearly Teddy was born in the nineteenth century (otherwise he would be as old as his own grandson!). The first two digits in his year of birth are 18, and the last two must add up to 132 (since he lived through a "turn of the century"). Hence he is 66, and was born in 1866. Teddy was fifty years old when Franklin was born.

F87S14

Let $m\angle A = 1$, $m\angle B = 2$, $m\angle C = 3$, $m\angle D = 4$ (all in radian measure). Since $0 < 1 < \pi/2 < 2 < 3 < \pi < 4$, angles B and C are in the second quadrant, angle A is in the first quadrant, and angle D is in the third quadrant. Hence sin4 = sinD < 0,and this cannot be the largest.
Next we compare sinB with sinC, by relating each to an angle in the first quadrant. We write $\sin 2 = \sin B = \sin(\pi - 2)$, $\sin 3 = \sin C = \sin(\pi - 3)$. Both $\pi - 2$ and $\pi - 3$ are less than $\pi/2$, and $\pi - 2 > \pi - 3$. Since the sine function increases through the first quadrant, $\sin(\pi - 2) = \sin 2 > \sin(\pi - 3) = \sin 3$, and the latter cannot be the largest.
It remains to compare sin1 with sin2, or $\pi - 2$ with 1. This can be done by noting that $\pi > 3$, so $\pi - 2 > 1$, and $\sin 2 = \sin(\pi - 2) > \sin 1$.
The above argument can be followed knowing only that $3 < \pi < 4$.

F87S15

Using absolute value for area, $|YBC| = (1/5)|ABC|$. Since these two triangles have equal altitudes from point B, AC:YC = 5:1, so YC = 6, AY = 24.
We now use the same argument with triangles AQY, XQY. Since $|XQY| = (1/3)|AQY|$, and the two triangles have the same altitude from point Q, XY = (1/3)AY = 8.

F87S16

We compute successively a^2, a^3, a^4, and a^5:
$a^2x^2 + 1/x^2 + 2$, or $x^2 + 1/x^2 = a^2 - 2$.

$$a(a^2 - 2) = a^3 - 2a = (x^2 + 1/x^2)(x + 1/x)$$
$$= x^3 + 1/x^3 + x + 1/x = x^3 + 1/3x^3 + a,$$

which can be written as $x^3 + 1/x^3 = a^3 - 3a$.
$(a^2 - 2)^2 = a^4 - 4a^2 + 4 = (x^2 + 1/x^2)^2 = x^4 + 1/x^4 + 2$, which can be written as $x^4 + 1/x^4 = a^4 - 4a^2 + 2$.
Finally,

$$\left(x^4 + 1/x^4\right)\left(x + 1/x\right) = x^5 + 1/x^5 + x^3 + 1/x^3$$
$$= a\left(a^4 - 4a^2 + 2\right) = a^5 - 4a^3 + 2a$$

so $x^5 + 1/x^5 = \left(a^5 - 4a^3 + 2a\right) - \left(a^3 - 3a\right) = a^5 - 5a^3 + 5a$.

F87S17

$\log_3\left(\log_2 x\right) = 4$, so $\log_2 x = 81$, and $x = 2^{81}$.

F87S18

For any M, quadrilateral ANMP is cyclic. Since NP is a chord whose inscribed angle is constant (it's always angle A), the length of NP equals $d \sin A$ (see diagram), where d is the diameter of the circle. Then the shortest value of NP corresponds to the smallest circle through ANMP. But the diameter of this circle is AM, so we need to minimize AM. This means that AM must be perpendicular to BC.

To find the ratio $M_oB : M_oC$, we let $BM_o = p$, $CM_o = q$, $AM = x$, and solve simultaneously the equations:

$$p^2 + x^2 = 49$$
$$q^2 + x^2 = 64$$

Subtracting, we find $q^2 - p^2 = 15 = \left(q + p\right)\left(q - p\right)$, and since $q + p = 5$, we find $q - p = 3$. This leads to $q = 4$, $p = 1$, and $p : q = 1:4$.

Fall 1987 – Senior A – 4 – Solutions

F87S19

The largest number in the product clearly cannot be more than 1000, and if 1000 is included in the product, the rest of the natural numbers must all be 1's. This product yields a sum of 1999. If the largest number in the product is less than 1000, it must be at most 500. Replacing the 1000 in the product by 500, and one of the 1's by a two decreases the sum, and if we make further replacements, we will clearly decrease the sum further. Hence the maximum sum is 1999.

F87S20

Let $\angle BAD$ be an acute angle of the rhombus. The area of the rhombus can be computed as $(1/2)AC \cdot BD$, or as $AB^2 \sin \angle BAD$. But the problem states that $AC \cdot BD = AB^2$, so that $\sin \angle BAD = 1/2$ and $\angle BAD = 30°$.

F87S21

There are six distances between the four poles, so the largest amount necessary is $6. If Ajax measures three sides and the diagonals, for a total fee of $5, Freddy can use the converse of the Pythagorean theorem to find out if the sides meet at right angles and if one pair of opposite sides are equal. Then it is easy to show that the plot is rectangular. But if Ajax measures only four lines, Freddy may not have enough information. Certainly, if opposite sides, or two diagonals are not equal, he will find out quickly that his plot is not rectangular. But Ajax may measure as many as four lines and still not be able to reach a conclusion:

a) If they measure only one diagonal, the second diagonal might be larger than the first. Even if the pair of opposite sides measured are equal, the figure may or may not be rectangular.
b) If they measure both diagonals, and two adjacent sides, the diagonals may or may not bisect each other (see diagram).

[diagram goes here]

c) If they measure both diagonals an a pair of opposite sides, the figure could be a rectangle or an isosceles trapezoid.

F87S22

The given relationship is equal to:

$ac^2 + bc^2 + c^3 = a^3 + b^3 + c^3$, or $ac^2 + bc^2 = a^3 + b^3$,

or $c^2 = (a^3 + b^3)/(a + b) = a^2 - ab + b^2$. But $c^2 = a^2 - 2ab\cos C + b^2$ (by the law of cosines), so $\cos C = 1/2$, and $C = 60°$.

F87S23

The given equation is equivalent to $\left[x^2+4x-5\right]\left[x^2+4x-21\right]=297$. Letting $y=x^2+4x-5$, this can be written as $y(y-16)=297$, or $y^2-16y-297=(y-27)(y+11)=0$, and $y=27$, -11. These two cases yield:

(i) $x^2+4x-5=27$; $x^2+4x-32=(x+8)(x-4)=0$, and $x=-8$, $+4$

(ii) $x^2+4x-5=-11$; $x^2+4x+6=0$, and $x=-2\pm i\sqrt{2}$.

F87S24

Suppose that the small circle has rotated through an angle $\theta=\angle QXP'$ (see diagram). To roll without slipping, $\overset{\frown}{QP'}$ must be equal $\overset{\frown}{QP}$, or $\angle QXP'$ must be double angle QOP (since the small radius is half the large radius). But $\angle QOP'=(1/2)\overset{\frown}{QP'}=(1/2)\angle QXP'=\angle QOP$, so O, P' and P are collinear. The required locus is simply a diameter of the large circle.

Fall 1987 – Senior A – 5 – Solutions

F87S25

We can express 24 as a product in these ways:

Factors	Sum	Factors	Sum
2, 12	14	2, 2, 6	10
3, 8	11	2, 3, 4	9
4, 6	10	2, 2, 2, 3	9

The largest sum is 14.
What's going on here?

F87S26

In the diagram at right, $p^2 + x^2 = n^2 - 2n + 1$, and $q^2 + x^2 = n^2 + 2n + 1$. Subtracting, we find that $q^2 - p^2 = (q + p)(q - p) = 4n$, or, since $q + p = n$, $q - p = 4$.

F87S27

Letting x = 2, we find f(2)f(3) = 3, so f(3) = 3/5. Letting x = 3, we find f(3)f(4) = 3, so f(4) = 3/(3/5) = 5. Continuing thus, we find f(5) 3/5, f(6) = 5, … f(10) = 5.

F87S28

We use absolute value for area. From similar triangles PYB, QYC (see diagram), PY:YC = BY:YQ = PB:BQ = 2:1 = $|BYP|:|BYC| = |BYC|:|QYC|$. Hence,
$|QYC| = (1/3)|QBC| = 2/3$, and $|PBC| = 4$, so $|PYQCB| = 4 + 2/3 = 14/3$.
From similar triangles AXP, DXQ, AX:XQ = PX:XD = 2:3, so
$|DXQ| = 3/5|ADQ| = (3/5)6 = 18/5$. Now $|APD| = 4$, so $|ADQXP| = 4 + 18/5 = 38/5$.
Hence

$$\begin{aligned}|XPYQ| &= |ABCD| - |PYQCB| - |ADQXP| \\ &= 16 - 14/3 - 38/5 \\ &= 240/15 - 70/15 - 114/15 \\ &= 56/15\end{aligned}.$$

F87S29

Let $y = x^2 + 2x - 3$. Then ..

$$\frac{1}{y} + \frac{18}{y+5} = \frac{18}{y+4}, \text{ or } \frac{19y+5}{y^2+5y} = \frac{18}{y+4}.$$

This leads to $y^2 - 9y + 20 = 0$, and $y = 5$ or 4.
The resulting equations in x are $x^2 + 2x - 3 = 5$, which has roots 2 and –4, and $x^2 + 2x - 3 = 4$, which has roots $-1 \pm 2\sqrt{2}$.

F87S30

We will show that the maximum volume of such a tetrahedron is achieved if two pairs of points are chosen, each pair at the endpoints of a diameter of one of the bases of the cylinder, and such that the diameters chosen are perpendicular (but in parallel planes).
To see this, we choose any four non-collinear points on the surface of the cylinder, and show that if they are not in the position described, then the volume of the tetrahedron they determine is not maximal.
Choose four such points, and think of any three of them as the base of the tetrahedron, and the fourth as the vertex. Call the base ABC and the vertex D. We can pass a plane through D parallel to the plane of ABC, and the distance between the planes is the altitude of the tetrahedron (from point D). If point D is not on the circumference of one of the bases of the cylinder, then we can displace the plane of D parallel to itself, until its intersection with the cylinder is a single such point. Such displacement holds the base of the tetrahedron constant, and increases its altitude. Therefore the displacement will increase its volume as well. This shows that is D is not in the circumference of a base, then the volume of ABCD is not maximal.
Thus we may assume that point D is on the circumference of a base of the cylinder. Next we turn to points A, B, and C. If none of these are on the circumference of a base, we can displace the plane ABC parallel to itself, an away from the plane through D, so that triangle ABC remains congruent to its original shape, and the distance between the two planes increases. We can do this so long as none of the points A, B, or C are on the circumference of the opposite base of the cylinder, and such displacement again increases the altitude (and the volume) of ABCD. Hence we can assume that another vertex of the tetrahedron, say A, is on the opposite base of the cylinder from point D.
We now show that the same must be true of the other two vertices of the tetrahedron. Assume that neither of B or C is on a base of the cylinder. We may choose one of the points (call it point C) such that the plane ACD is not parallel to the axis of the cylinder. (If both planes ADB and ADC were parallel to the axis, they would have to be identical, since they both contain line AD. Then the points A, B, C, and D would be coplanar, and would not form a tetrahedron.) Now think of ADB as the base of the tetrahedron, with C as its vertex. Passing a plane through C parallel to ADB, we can once more displace this plane, parallel to itself, until its intersection with the cylinder is a single point. This displacement increases the volume of the tetrahedron, and can be continued so long as point C is not on the circumference of a base. When C reaches such a position, it is diametrically opposite to point A.
A similar argument using ACD as the base of the tetrahedron, and point B as its vertex, shows that B must be on the circumference of the base opposite to the base containing point C, and diametrically opposite to D.
Thus the tetrahedron of maximal volume must have diameters of opposite bases of the cylinder as one pair of opposite edges. An argument similar to those given above shows that these two diameters must be perpendicular.
It remains to compute the volume of this tetrahedron. The diagram shows a cross section of the cylinder through points A, C, and D (the center of the opposite base). Points B and D are on a line through O perpendicular to the plane of the diagram. Note that $s^2 = h^2 + r^2$, that AO = s is the altitude of triangle ABD, and that if CBD is chosen as the base of the tetrahedron, the altitude of the tetrahedron is the segment marked x.

Computing the area of ACO in two ways, we find that $xs = 2rh$, so $x = 2rh/s$. The area of CBD is $(1/2)2rs = rs$, so the volume of the tetrahedron is $(1/3)(rs)(2rh/s) = (2/3)r^2h$. In the present case, this is 50.

[diagram goes here]