New York City
Interscholastic Mathematics League

SENIOR B DIVISION

Part I   Time: 11 Minutes   Contest Number One   Spring, 1987

S87B1. Sesame Street runs for 25 blocks. There is a single street sign at each intersection of Sesame Street with another street, including the beginning and end of Sesame Street. How many street signs are there which identify Sesame Street?

S87B2. In triangle ABC, AB = 6, AC = 7, and BC = 8. Points X and Y are chosen on side BC such that BA = BX and CA = CY. Compute the length of line segment XY.

Part II   Time: 11 Minutes   NYCIML Contest Number One   Spring, 1987

S87B3. If a, b, c, and d are consecutive positive integers, and $\frac{a}{b} \cdot \frac{c}{b} = \frac{c}{d}$, compute d.

S87B4. In a certain isosceles triangle, the median drawn to the base is equal to half the base. Compute the sine of the vertex angle of this triangle.

Part III   Time: 11 Minutes   NYCIML Contest Number One   Spring, 1987

S87B5. Find all real numbers x such that x - 1 is the reciprocal (multiplicative inverse) of x + 1.

S87B6. Find a two-digit number (in decimal notation) such that the sum of the digits is 12, and if the digits are reversed, the number is increased by 75%.

ANSWERS

1. 26
2. 5
3. 4
4. 1
5. $\pm\sqrt{2}$ (both values required)
6. 48

New York City Interscholastic Mathematics League

SENIOR B DIVISION

Part I   Time: 11 Minutes   Contest Number Two   Spring, 1987

S87B7. Mary biked to the park at a constant rate of 10 miles per hour, and rode back by bus at a rate of 40 miles per hour. Sue rode her motorcycle to the park and back, along the same route as Mary, and her ride took the the same amount of time. If Sue rode at a constant rate, compute this rate in miles per hour.

S87B8. On hypotenuse AB of right triangle ABC, point D is chosen so that CB = BD. If $m\angle B = 40^\circ$, compute the degree-measure of angle ACD.

Part II   Time: 11 Minutes   NYCIML Contest Number Two   Spring, 1987

S87B9. Find the least positive integer which is greater than 1 and which leaves a remainder of 1 when divided by 12 and by 42.

S87B10. In triangle PQR, $m\angle P = 80^\circ$ and $m\angle Q = 10^\circ$. Median RM is 10 inches in length. From point M, perpendiculars MX and MY are drawn to PR and QR respectively. Compute the length of line segment XY.

Part III   Time: 11 Minutes   NYCIML Contest Number Two   Spring, 1987

S87B11. How many integers x are there such that $1000 \le x \le 10000$, and x is a perfect cube (the cube of some other integer)?

S87B12. In a certain isosceles triangle, the length of the base and the median to one <u>leg</u> are equal. Compute the cosine of the vertex angle of this isosceles triangle.

ANSWERS

7. 16 or 16 MPH   9. 85   11. 12

8. 20 or 20° 10. 10   12. 3/4

New York City
Interscholastic Mathematics League

SENIOR B DIVISION

Part I Time: 11 Minutes Contest Number Three Spring, 1987

S87B13. Find all real numbers x such that

$$2^{\log_4 x} = 16.$$

S87B14. One side of an equilateral triangle is 12. The triangle is inscribed in a circle. Compute the length of a side of a square inscribed in this same circle.

Part II Time: 11 Minutes NYCIML Contest Number Three Spring, 1987

S87B15. The average of three numbers is 36. If two of the numbers are doubled, the average of the new set of numbers is 49. Find the third number.

S87B16. Al can choose one of three days to take a math test. He always fails such tests if it rains on the day he takes them, and he always passes such tests if it does not rain. Weather forecasts predict that the chance of rain is 1/3 on each of the three days Al can take the test. What is the probability that Al will be able to pass the test, if he waits for the best possible day to take it?

Part III Time: 11 Minutes NYCIML Contest Number Three Spring, 1987

S87B17. If $a + 1/a = 3$, compute the numerical value of $a^2 + 1/a^2$.

S87B18. On a rectangular coordinate system, point A has coordinates (3,8) and point B has coordinates (8,3). Point P has coordinates (k,0), and the sum AP + BP is as small as possible. Compute the numerical value of k.

ANSWERS

13. 256	15. 69	17. 7
14. $4\sqrt{6}$	16. 26/27	18. 73/11

New York City Interscholastic Mathematics League

SENIOR B DIVISION

Part I Time: 11 Minutes Contest Number Four Spring, 1987

S87B19. How many positive integers are divisors of the integer 392 (remember that 1 and 392 are both divisors of 392)?

S87B20. In an isosceles triangle, the length of the base is 30, and one of the legs is 17. Find the area of this isosceles triangle.

Part II Time: 11 Minutes NYCIML Contest Number Four Spring, 1987

S87B21. Find all integers x such that $(2/3)^x \cdot (9/8)^x = 27/64$.

S87B22. Find all real numbers x such that $|x + 3| + |x - 1| = 6$.

Part III Time: 11 Minutes NYCIML Contest Number Four Spring, 1987

S87B23. Find all real numbers x such that:

$$\sqrt{x/2} + \sqrt{2x/9} + \sqrt{x/8} = 1/12.$$

S87B24. If x is a real number such that $\sin x + \cos x = 1/5$, compute the numerical value of $\sin 2x$.

ANSWERS

19. 12

20. 120

21. 3

22. −4, 2: both required

23. 1/338

24. −24/25

New York City Interscholastic Mathematics League

SENIOR B DIVISION

Part I   Time: 11 Minutes   Contest Number Five   Spring, 1987

S87B25. In an isosceles triangle, the length of base AC is 30, and the altitude to leg BC is 24. Compute the length of BC.

S87B26. Find all ordered pairs (x,y) of real numbers such that

$$x^{y-2} = 4, \text{ and } x^{2y-3} = 64.$$

Part II   Time: 11 Minutes   NYCIML Contest Number Five   Spring, 1987

S87B27. What is the average (arithmetic mean) of the degree-measures of the interior angles of a (convex) hexagon?

S87B28. Compute the smallest <u>positive integer</u> value of x such that $x^2 - 8x - 2 > 0$.

Part III   Time: 11 Minutes   NYCIML Contest Number Five   Spring, 1987

S87B29. Find all real numbers x such that $\log_x 2 + \log_2 x = 5/2$.

S87B30. The integer 999,999,995,904 can be factored as $a^{16} \cdot b^2 \cdot c \cdot d \cdot e \cdot f$, where a, b, c, d, e, and f are distinct prime numbers, and $a<b<c<d<e<f$. Compute the numerical value of f.

ANSWERS

25. 25

26. (4,3): ordered pair required

27. 120 or 120°

28. 9

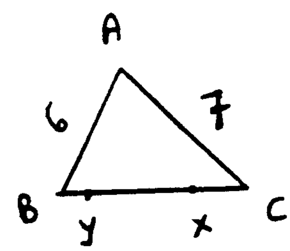
29. 4, $\sqrt{2}$: both required

30. 601

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE
Spring, 1987
Senior B Division Contest Number One

S87B1. There are 26 streets dividing the 25 blocks that Sesame Street runs, so there are 26 street signs identifying Sesame Street.

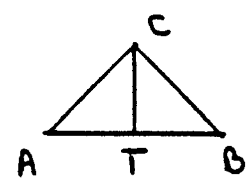
S87B2. We have $XC = BC - BX = BC - BA = 8 - 6 = 2$,
$BY = BC - YC = BC - AC = 8 - 7 = 1$.
Hence $XY = BC - BY - XC = 8 - 3 = 5$.



S87B3. Let $a = x$, $b = x+1$ $c = X+2$, $d = x+3$. Then
$\frac{x}{x+1} \cdot \frac{x+2}{x+1} = \frac{x+2}{x+3}$, or

$x(x+2)(x+3) = (x+1)^2(x+2)$. If $x = -2$, the numbers a, b, c, d are not positive integers. Hence $x + 2 \neq 0$, and we can divide by this quantity, to get $x(x+3) = (x+1)^2$
$x^2 + 3x = x^2 + 2x + 1$, and $x = 1$. The numbers a, b, c, d are 1, 2, 3, and 4, and $d = 4$.

S87B4. Since $CT = AT = TB$, points A, C, and B are on a semicircle with AB as diameter. This makes the triangle a right isosceles triangle, with $\angle C = 90°$. The sine of 90° is 1.



S87B5. We have $(x-1)(x+1) = 1$, or $x^2 - 1 = 1$, or $x^2 = 2$, and $x = \pm\sqrt{2}$.

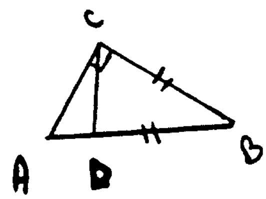
S87B6. If the tens digit is t, and the units digit is u, then $t + u = 12$, and $10u + t = (7/4)(10t + u)$. Solving simultaneously, we find $u = 8$ and $t = 4$. The original number is 48.

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE
Spring, 1987
Senior B Division Contest Number Two

S87B7. If the distance to work is d, then the woman spent d/10 hours travelling to work, and d/40 hours travelling from work. The total time spent travelling is then d/10 + d/40. The average rate for the round trip is the total distance divided by the total time, or (2d)/(d/10 + d/40) = 80d/5d = 80/5 = 16 miles per hour.

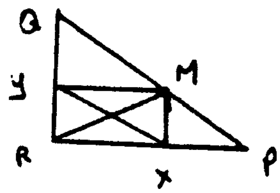
Note that the average rate does not depend on the distance travelled: this rate is called the <u>harmonic mean</u> of the two original rates.

S87B8. In isosceles triangle BCD (see diagram), $\angle BDC = \angle BCD = 70^\circ$, so $\angle ACD = 90-70 = 20^\circ$.



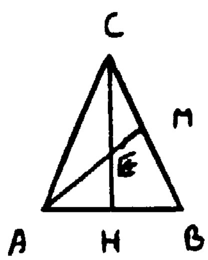
S87B9. If the required integer is N, then N-1 is the least common multiple of 12 and 42. Since $12 = 2^2 \cdot 3$ and $42 = 2 \cdot 3 \cdot 7$, the least common multiple is $2^2 \cdot 3 \cdot 7 = 84$, and N = 85.

S87B10. Since $\angle QRP = 90^\circ$, quadrilateral RXMY is a rectangle. Hence diagonals MR, XY are equal, and XY = 10.



S87B11. First note that $1000 = 10^3$, so this is the first cube in the given range. Since $21^3 = 9261$ and $22^3 = 10648$, 21^3 is the last cube in the given range. Thus there are 21 - 10 + 1 = 12 such cubes.

S87B12. If the triangle is ABC, we draw altitude CH. Then AH = AB/2. Since the altitude to the base is also a median, and since the medians of a triangle divide each other in the ratio 2:1, AE = (2/3)AM = (2/3)AB. Finally, $\angle ACB = \angle MAB$ (since triangle ABC, ABM are similar), and $\cos \angle ACB = \cos \angle MAB = \cos \angle EAH$ = AH:AE. This ratio is (1/2)AB:(2/3)AB = 1/2 : 2/3 = 3/4.



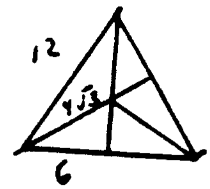
NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE
Spring, 1987
Senior B Division Contest Number Three

S87B13. We have:

$$2^{\log_4 x} = 16 = 2^4, \text{ so } \log_4 x = 4, \text{ and}$$

$$x = 4^4 = 256.$$

S87B14. The center of the circle circumscribing an equilateral triangle is also the intersection of its altitudes. By examining the 30-60-90 triangles in the figure, we find that this radius is $4\sqrt{3}$. The side of a square inscribed in a circle of radius r is $r\sqrt{2}$, which here is $4\sqrt{2}\sqrt{3} = 4\sqrt{6}$.



S87B15. The sum of the three numbers is three times their average. Hence, if the numbers are a, b, and c, then $a + b + c = 36 \cdot 3 = 108$, and from the conditions of the problem, we find $2a + 2b + c = 49 \cdot 3 = 147$. Doubling the first equation and subtracting the second gives $c = 69$.

Note that the first two numbers are not uniquely determined.

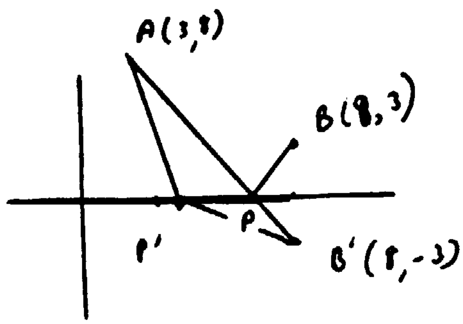
S87B16. The probability that it will rain all three days, and hence that Al will fail the exam no matter what he does, is $(1/3)^3 = 1/27$. Hence the probability that it will not rain on at least one day is 26/ 27. If Al waits for the first fair day, this will be the probability that he will pass.

S87B17. We have $(a + 1/a)^2 = 9 = a^2 + 2 + 1/a^2$, so $a^2 + 1/a^2 = 7$.

S87B18. If $B' = (8,-3)$ is the reflection of B in the x-axis, then A, P and B' must be collinear. Thus the slope of line APB' must be constant, or

$$\frac{8-0}{3-k} = \frac{0-(-3)}{k-8}, \text{ or } 9 - 3k = 8k - 64, \text{ and } k = 73/11.$$

To see that P is indeed minimal, select any point P' on the x-axis, and note that $AP' + P'B = AP' + P'B' > AB'$, since the sum of two sides of a triangle is greater than the third side.



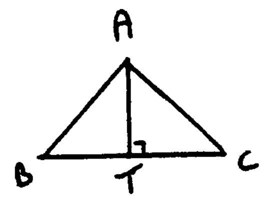
NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE
Spring, 1987
Senior B Division Contest Number Four

S87B19. The number 392 factors into primes as $2^3 \cdot 7^2$. Any factor of 392 must be made up of fewer than four factors of 2 and fewer than three factors of 7, and such a description uniquely determines each divisor. Thus in forming a divisor, there are four possibilities for the number of factors to 2 to contribute, and 3 possibilities for the number of factors of 7, making 12 possiblities in all.

S87B20. In right triangle ABT, hypotenuse AB = 17, and BT = BC/2 = 15, so $AT^2 = AB^2 - BT^2 = 289 - 225 = 64$, and AT = 8. Then the area of the triangle is $8 \cdot 30/2 = 120$.

The value of AT could also be found by noticing the Pythagorean triple 8-15-17 in triangle ABT.

S87B21. We have:



$$\frac{2^x}{3^x} \cdot \frac{3^{2x}}{2^{3x}} = \frac{27}{64}\text{ , or}$$

$$\frac{3^x}{2^{2x}} = \frac{27}{64}\text{ , and } 64 \cdot 3^x = 27 \cdot 2^x\text{, so that } x = 3.$$

S87B22. We distinguish three cases:

(i) For $x > 1$, we have $x+3 + x-1 = 6$, or $2x + 2 = 6$, and $x = 2$.

(ii) For $-3 < x < 1$, we have $x+3 + 1-x = 6$, which has no solution.

(iii) For $x < -3$, we have $-x-3 + 1-x = 6$, or $-2x - 2 = 6$, and $x = -4$.

S87B23. The given equation can be written as

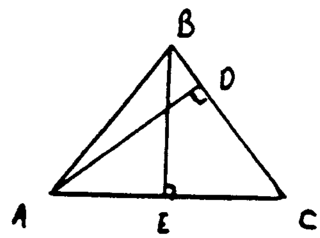
$\sqrt{x} \cdot \sqrt{2}/2 + \sqrt{x} \cdot \sqrt{2}/3 + \sqrt{x} \cdot \sqrt{2}/4 = 1/12$, or

$6\sqrt{2x} + 4\sqrt{2x} + 3\sqrt{2x} = 1$, or $13\sqrt{2x} = 1$, and $\sqrt{2x} = 1/13$, so $x = 1/338$.

S87B24. Squaring the given relationship gives:
$\sin^2 x + 2 \sin x \cos x + \cos^2 x = 1/25$,
or $1 + 2 \sin x \cos x = 1/25$, and $2 \sin x \cos x = \sin 2x = -24/25$.

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE
Spring, 1987
Senior B Division Contest Number Five

S87B25. The Pythagorean theorem in triangle ADC (see diagram) shows that DC = 18. Then, from similar triangle DAC, EBC, DC:AC = EC:BC, or 18/30 = 15/BC, and BC = 25.



S87B26. Taking logarithms to the base two of each equation, we have

$$(y-2)\log_2 x = 2, \text{ and } (2y-3)\log_2 x = 6.$$

Letting $z = \log_2 x$, and dividing the first equation by the second, we find:

$\dfrac{y-2}{2y-3} = \dfrac{1}{3}$, or $3y - 6 = 2y - 3$, and $y = 3$. Substituting shows that $x = 4$.

S87B27. Since the sum of the interior angles of a hexagon is constant, their average will be the value they would take on if they were all equal. This is the size of an interior angle of a regular hexagon, or 120°.

S87B28. The quadratic formula shows that the roots of the given equation are $4 \pm 3\sqrt{2}$. Since $3\sqrt{2}$ is about 4.242, the roots of the equation are approximately 8.242, and -0.242.

Now the graph of the function $y = x^2 - 8x - 2$ is a parabola with its vertex pointing down, and the roots are the x-intercepts. Hence the value of the function will be positive <u>outside</u> of the interval determined by the roots. The smallest positive integer in this region is 9.

S87B29. Since $\log_x 2 = 1/\log_2 x$, we can let $u = \log_2 x$, and $u + 1/u = 2 + 1/2$. By inspection, $u = 2, 1/2$, and there can be no more than these two solutions to the quadratic equation in u.

Then $\log_2 x = 2$, and $x = 4$, or $\log_2 x = 1/2$, and $x = \sqrt{2}$.

S87B30. The given integer is slightly less than 10^{12}, and is in fact equal to $10^{12} - 4096 = 10^{12} - 2^{12}$. This number factors as:

$2^{12}(5^{12} - 1)$, and

$$\begin{aligned} 5^{12} - 1 &= (5^6-1)(5^6+1) \\ &= (5^3-1)(5^3+1)(5^2+1)(5^4-5^2+1) \\ &= (5-1)(5^2+5+1)(5+1)(5^2-5+1)(5^2+1)(5^4-5^2+1) \\ &= 2^2\cdot 31\cdot 2\cdot 3\cdot 3\cdot 7\cdot 2\cdot 13\cdot 601. \end{aligned}$$

hence the given integer factors as $2^{16}\cdot 3^2\cdot 7\cdot 13\cdot 31\cdot 601$, and the largest prime factor is 601.