New York City Interscholastic Mathematics League

Part I Time: 11 Minutes Contest Number One Spring, 1987

S87S1. If x is a decimal digit, and the decimal numeral 19x83 represents a perfect cube, find x.

S87S2. On a grand tour of Europe, Harry and Ruth spent the first Tuesday of one month in London, and the first Tuesday after the first Monday of that month in Paris. The next month, they spent the first Tuesday in Madrid, and the first Tuesday after the first Monday in Rome. In what month were Harry and Ruth in London? (Spell out the month in giving your answer.)

Part II Time: 11 Minutes NYCIML Contest Number One Spring, 1987

S87S3. Compute $\log_{1/8} \sin 4350^\circ$.

S87S4. In the state lottery, half the tickets will win some kind of prize. Mr. Reuterman, a math teacher, sells bundles of tickets at random, each having the same number of tickets. He claims that if you buy one of his bundles, the probability of getting at least one winning ticket is greater than .95. At least how many tickets must be in his bundles?

Part III Time: 11 Minutes NYCIML Contest Number One Spring, 1987

S87S5. Find all real numbers x such that $2^{2x} - 2^x = 992$.

S87S6. The sides of a certain triangle can be represented by $x^2 + x + 1$, $2x + 1$, and $x^2 - 1$. Find the degree-measure of the largest angle of this triangle.

ANSWERS

1. 6
2. February
3. 1/3
4. 5
5. 5
6. 120 or 120°

New York City
Interscholastic Mathematics League

Part I   Time: 11 Minutes   Contest Number Two   Spring, 1987

S87S7. Find all real numbers x such that
$$|3x - 7| = x + 5.$$

S87S8. Two machines are doing a certain job. The first machine worked for 5 hours alone, and the second for two hours alone. After this, they had completed half the job. They completed another 1/4 of the job by working together for an hour and a half. The remaining quarter of the job was completed by the second machine working alone. How long did this second machine have to work alone to complete the last quarter of the job?

Part II   Time: 11 Minutes   NYCIML Contest Number Two   Spring, 1987

S87S9. If $f(x) = \cos x + |\cos x|$, for how many values of x is $0 \le x \le 2\pi$ and $f(x) = 1$?

S87S10. In quadrilateral ABCD, $AB \perp BC$, $AD \perp DC$, angle BAD = 105°, AB = 2, and CD = $2\sqrt{2}$. Compute BD (answer may be left in radical form).

Part III   Time: 11 Minutes   NYCIML Contest Number Two   Spring, 1987

S87S11. An urn contains twelve marbles, of which five are white and the rest black. Three marbles are chosen in succession, without replacement. What is the probability that at least one of these is black?

S87S12. The function f(x) is defined for all natural numbers by
$$f(1) = 1 \text{ and } f(x+y) = \frac{f(x)\,f(y)}{f(x)+f(y)}$$
Find f(1987).

ANSWERS

7. 6, 1/2 : both required

8. 9/4

9. 2

10. $2\sqrt{2+2\sqrt{3}}$ or $\sqrt{2}+\sqrt{6}$

11. 21/22

12. 1/1987

New York City
Interscholastic Mathematics League

Part I Time: 11 Minutes Contest Number Three Spring, 1987

S87S13. In a certain math class there are 5 boys and 12 girls. Each boy had an equal number of pencils, and the girls had no pencils at all. Then each boy gave a pencil to each girl. After this, all the students in the class had the same number of pencils. How many pencils did each boy have to begin with?

S87S14. Points E, F, and G are midpoints of three adjacent edges of a cube. If an edge of the cube is 4, find the area of triangle EFG.

Part II Time: 11 Minutes NYCIML Contest Number Three Spring, 1987

S87S15. Find the radian measure of the smallest positive x such that

$$16^{\cos\frac{x}{3}} - 4^{\cos\frac{x}{3}} = 2$$

S87S16. In triangle ABC, BT and CT bisect angles ABC, ACB respectively. Points M on AB and N on AC are chosen such that MN is parallel to BC and point T is between M and N. If MB = 6 and NC = 8, compute the length of MN.

Part III Time: 11 Minutes NYCIML Contest Number Three Spring, 1987

S87S17. Find the largest real value of c such that the equation

$\sqrt{x/4+2} = c + \sqrt{x/4-3}$ has at least one <u>real</u> solution for x.

S87S18. If the real numbers x, y, and z satisfy $x^{20} + y^{20} = z^{20}$, find the maximum possible value of $(xy/z^2)^{20}$.

ANSWERS

13. 17
15. π
17. $\sqrt{5}$
4. $2\sqrt{3}$
16. 14
18. 1/4

New York City Interscholastic Mathematics League

Part I Time: 11 Minutes Contest Number Four Spring, 1987

S87S19. Find all real numbers x such that $3^x = 3 + 2(\sqrt{3})^x$.

S87S20. In parallelogram ABCD, the bisectors of interior angles ABC and BCD intersect on side AD, between points A and D. Compute the ratio AB:BC.

Part II Time: 11 Minutes NYCIML Contest Number Four Spring, 1987

S87S21. A polyhedron whose faces are all convex polygons has five vertices. What is the __smallest__ number of faces that this polyhedron can have? (You may wish to use Euler's formula for polyhedra, which says that V + F = E + 2, where V, F, and E are respectively the number of faces, vertices, and edges of such a polyhedron.)

S87S22. Find all ordered pairs (x,y) of real numbers such that

$$x^2 + xy = 3, \text{ and}$$
$$y^2 + xy = 6.$$

Part III Time: 11 Minutes NYCIML Contest Number Four Spring, 1987

S87S23. Two hikers left an inn to climb a mountain. Their trip up and down the mountain took six hours altogether. They travelled the same trail going up and down the mountain. The trail, in each direction, sometimes went uphill, sometimes downhill, and sometimes on level ground. The hikers' speed going uphill was 3 miles per hour. Going downhill it was 6 miles per hour, and on level gound it was 4 miles per hour. How long, in miles, was their entire round trip?

S87S24. Find the unique integer x such that $x^4 + 8x^3 + 20x^2 + 12x + 12$ is a perfect square.

ANSWERS

19. 2	20. 1/2	21. 5
22. (1,2);(-1,-2): both ordered pairs required	23. 12 or 12 miles	24. 2

New York City
Interscholastic Mathematics League

Part I Time: 11 Minutes Contest Number Five Spring, 1987

S87S25. An old-fashioned toaster has room for exactly two slices of bread, and it can toast only one side of each slice at a time. It takes 30 seconds to toast one side of a slice of bread. What is the smallest number of seconds in which three slices of bread can be toasted (on both sides)?

S87S26. If

$$a = \sqrt{1986} + \sqrt{1988}$$
$$b = 2\sqrt[4]{1986 \cdot 1988}$$
$$c = 2\sqrt{1987},$$

arrange the numbers a, b, c in <u>increasing</u> order.

Part II Time: 11 Minutes NYCIML Contest Number Five Spring, 1987

S87S27. Ms. Kaiser wants to select five students at random from her class of five girls and ten boys. What is the probability that exactly two of the students she selects will be girls?

S87S28. The surface area of a sphere, in square units, is $A\pi$, while its volume, in cubic units, is $B\pi$, where A and B are natural numbers between 1000 and 10000. Find the radius of the sphere.

Part III Time: 11 Minutes NYCIML Contest Number Five Spring, 1987

S87S29. Find the smallest integer which is larger than the sum

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{49^2} + \frac{1}{50^2}$$

S87S30. The radius of a sphere is 20 units. On the surface of the sphere are drawn three circles, each tangent externally to the other two. The radius of each of the circles is 10. A smaller fourth circle is tangent externally to each of the other three. Compute the radius of this fourth circle.

ANSWERS

25. 90

26. b, a, c

27. 400/1001

28. 18

29. 2

30. $10 - 10\sqrt{2/3}$ or $10 - \frac{10}{3}\sqrt{6}$ or equivalent

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE
Spring, 1987
Senior A Division Contest Number One

SOLUTIONS

S87S1. We have $8000 = 20^3 < 19x83 < 27000 = 30^3$, and if N^3 ends in the digit 3, N must end in the digit 7. Indeed, $27^3 = 19683$, and x = 6.

S87S2. Both months begin on a Tuesday, which can only happen if the number of days in the first month is a multiple of 7. This can only happen if the first month is February (of a non-leap year).

S87S3. We have $4350 = 12 \cdot 360 + 30$, so $\sin 4350^\circ = \sin 30^\circ = 1/2$. Since $(1/2)^3 = 1/8$, the given expression equals $\log_{1/8}(1/2) = 1/3$.

S87S4. If each bundle contains n tickets, then the probability of all of them being losing tickets is $1/2^n$. This must be less than .05. By direct computation, we find that $1/2^4 = 1/16 = .0625 > .05$, while $1/2^5 = 1/32 = 0.03125 < .05$, so there must be at least five tickets.

S87S5. Let $y = 2^x$. Then $y^2 = 2^{2x}$, and $y^2 - y = 992$. Solving this quadratic, we find $y^2 - y - 992 = (y-32)(y+31) = 0$, and y = 32 or -31. The second root is extraneous, so $y = 32 = 2^5$, and x = 5.

S87S6. If $a = x^2 + x + 1$, $b = 2x + 1$, $c = x^2 - 1$, then $\cos A = (b^2+c^2-a^2)/2bc$, and substituting the expressions in x for a, b, and c, we find that

$$\cos A = (-2x^3-x^2+2x+1)/(4x^3+2x^2-4x-2) = -1/2, \text{ so } A = 120^\circ.$$

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE
Spring, 1987
Senior A Division Contest Number Two

SOLUTIONS

S87S7. If $|3x - 7| = x + 5$, then $|x - 7/3| = x/3 + 5/3$. We distinguish two cases:

(i) $x > 7/3$: $x - 7/3 = x/3 + 5/3$, and $x = 6$.
(ii) $x < 7/3$: $7/3 - x = x/3 + 5/3$, and $x = 1/2$.

S87S8. Suppose the first machine alone would need a hours to do the job, and the second machine alone would need b hours to do the job. Then we have:

$$5/a + 2/b = 1/2$$
$$3/2a + 3/2b = 1/4.$$

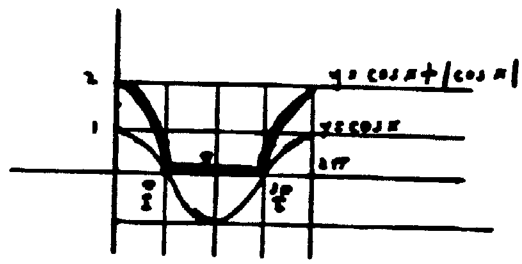
We need to find x, where $x/b = 1/4$.

Multiplying the first equation by 2ab and the second by 4ab, we find:

$$4a + 10b = ab = 6a + 6b, \text{ which leads to } a = 2b.$$

Then $5/2b + 2/b = 1/2$, or $9/2b = 1/2$, and $9/4b = 1/4$, so $x = 9/4$.

S87S9. This problem is most easily done graphically. If cos x is positive, the given function is equal to 2 cos x. If cos x is not positive, the given function is zero:



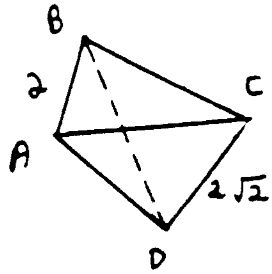
Thus there are two solutions, which occur when $\cos x = 1/2$, or $x = \pi/3$ and $4\pi/3$

S87S10. If we draw diagonal AC, and note that $105 = 60 + 45$, we can conjecture that $m\angle BAC = 60$ and $m\angle CAD = 45$. If this is the case, then, from 30-60-90 triangle ABC, and isosceles right triangle CAD, $CD = AD = 2\sqrt{2}$, and $AC = 4$. Since the values for AC agree, and since triangle ABD is determined by the given information, our conjecture about the angles must be correct.

Now we use the fact that quadrilateral ABCD can be inscribed in a circle, and $BC = 2\sqrt{3}$. Then, from Ptolemy's theorem, $4BD = 4\sqrt{2} + 4\sqrt{6}$, and $BD = \sqrt{2} + \sqrt{6}$ (an alternative form of this answer is $2\sqrt{2+\sqrt{3}}$).

That this solution is unique can be shown by calculating AD and BC in terms of AC, using the law of cosines.

S87S10



S87S11. The probability that each ball withdrawn will be white is $(5/12)(4/11)(3/10) = 1/22$. Hence the probability that not all are white (i.e. that at least one is black) is $1 - 1/22 = 21/22$.

S87S12. We have:

$$\frac{1}{f(x+y)} = \frac{1}{f(x)} + \frac{1}{f(y)}$$

for x, y rational and positive. Hence, if y=x,

$$\frac{1}{f(2x)} = \frac{2}{f(x)}$$

so $f(2x) = (1/2)f(x)$.

Next we let $y = 2x$, so

$$\frac{1}{f(3x)} = \frac{1}{f(x)} + \frac{1}{f(2x)} = \frac{1}{f(x)} + \frac{2}{f(x)} = \frac{3}{f(x)}$$

and $f(3x) = (1/3)f(x)$.

It seems that $f(nx) = (1/n)f(x)$, for natural numbers n. We can prove this by an induction. We know the result is true for $n = 1,2,3$. Suppose it is true for $(n-1)$. Then

$$\frac{1}{f(nx)} = \frac{1}{f((n-1)x)} + \frac{1}{f(x)} = \frac{n-1}{f(x)} + \frac{1}{f(x)} = \frac{n}{f(x)}$$

and $f(nx) = (1/n)f(x)$. Then $f(n) = f(n)f(1) = (1/n)\cdot 1 = 1/n$.

As a footnote, we can show that if $x = a/b$, for a, b integers, then

$f(x) = f(a/b) = f(a\cdot 1/b) = (1/a)f(1/b)$ (since the argument above does not really depend on x being an integer),

and $f(1/b) = b$ (this can be shown by another induction). Hence $f(a/b) = b/a$, or $f(x) = 1/x$ for rational x as well.

An extension of this result to the real numbers would involve an assumption of continuity for the function.

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE
Spring, 1987
Senior A Division Contest Number Three

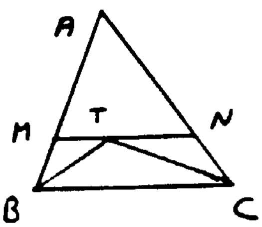
SOLUTIONS

S87S13. After the pencils were given, each girl had five pencils, so each boy also had five pencils. Since each boy gave away 12 pencils, each must have had 17 pencils to start with.

S87S14. The triangle EFG is equilateral, since the faces of the cube are all congruent. Its side is $\sqrt{8}$, so its area is $8\sqrt{3}/4 = 2\sqrt{3}$.

S87S15. See problem S87S5.
Let $y = 4^{\cos x/3}$. Then $y^2 - y = 2$, and $y = 2, -1$. Hence
$4^{\cos x/3} = 2$ (since this expression cannot equal -1), so $\cos x/3 =$
$= 1/2$, and $x/3 = \pi/3$, so $x = \pi$.

S87S16. Since MN is parallel to BC, <MBT = <MTB and <NTC = <NCT. Then triangles MBT, NCT are isosceles, so MT=MB=6 and NT=NC=8. Hence MN =
= MT + TN = 14.



S87S17. Let $y = \sqrt{x/4 - 3}$. Then $y^2 + 5 = c^2 + 2cy + y^2$, so
$c^2 + 2cy - 5 = 0$, and $y = (5-c^2)/2c$. For real solutions, this expression must be non-negative, so $c \leq \sqrt{5}$. If $c = \sqrt{5}$, we have $y = 0$ and $x = 12$.

S87S18. Squaring, we find $x^{40} + 2x^{20}y^{20} + y^{40} = z^{40}$, or

$$z^{40} = x^{40} + y^{40} + 2x^{20}y^{20}$$
$$= x^{40} - 2x^{20}y^{20} + y^{40} + 4x^{20}y^{20}$$
$$= (x^{20} - y^{20})^2 + 4x^{20}y^{20}, \text{ so}$$

$$z^{40}/x^{20}y^{20} = 4 + \frac{(x^{20} - y^{20})^2}{x^{20}y^{20}} \geq 4$$ (since an even power cannot be negative).

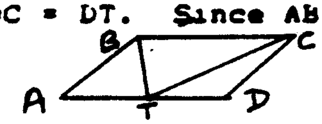
Hence $(xy/z^2)^{20} \leq 1/4$. Equality holds if $x = y$.

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE
Spring, 1987
Senior A Division Contest Number Four

SOLUTIONS

S87S19. Let $y = \sqrt{3}$. Then $(\sqrt{3})^{2x} = 3 + 2(\sqrt{3})^x$, or $y^2 - 2y - 3 = 0$, and $(y-3)(y+1)=0$, so $y = 3, -1$. The second root is extraneous, so $\sqrt{3}^x = 3$, and $x = 2$.

S87S20. Since AD is parallel to BC, <ABT = <TBC = <ATB. Similarly, <BCT = = < TCD = <DTC (compare problem S87S16). Hence triangles ABT, DCT are isosceles, and AB = AT, DC = DT. Since AB = DC, we have BC = AD = 2AB, and AB:BC = 1/2.



Note also that BT⊥CT.

S87S21. Using Euler's formula, we have $V+F = E+2$. Here, $V = 5$, so $E-F = 3$. Since there are only five vertices, the faces can only be triangles or quadrilaterals.

First let us suppose the faces are all triangles. Let us count the edges. There are three edges to each face, but this counts each edge twice: once for each face it bounds. Thus $E = 3F/2$. Substituting in the relationship $E-F=3$, we find $F = 6$. The solid composed of two regular tetrahedra with a common face is an example of this sort of figure.

Suppose now that all but one of the faces are triangles, and the anomolous face is quadrilateral. Then the polyhedron must be a pyramid with a quadrilateral base, so there are five faces.

If two or more of the faces are quadrilaterals, there must be at least six vertices in the polyhedron.

S87S22. Adding the two given equations, we find $x^2 + 2xy + y^2 = (x+y)^2 = 9$, and $x + y = \pm 3$. Subtracting the two equations, we find $y^2 - x^2 = 3$, so $(y+x)(y-x) = 3$. Substituting ± 3 for $(x+y)$ shows that $y-x = \pm 1$. This leads to two possible sets of simultaneous equations.

(i) $x+y = 3$
$y-x = 1$, and $(x,y) = (1,2)$

(ii) $x+y = -3$
$y-x = 1$, and $(x,y) = (-1,-2)$

S87S23. Each mile on level ground takes the hikers 1/4 hour to travel (each way). Hence, on the round trip, they cover each level mile in 1/2 hour.

Each mile uphill (on the way out) takes 1/3 hour on the way up and 1/6 hour on the way down (since an uphill on the way out will be a downhill on the way back). Hence each such mile takes 1/2 hour.

Similarly, each mile downhill (on the way out) takes 1/2 hour for the round trip. Hence, each mile of the round trip will take 1/2 hour. Since the hikers were gone 6 hours, they must have travelled 12 miles out and 12 back, or 24 miles altogether.

F87S24. The trick is to match the given expression as closely as possible with the square of a polynomial, and examine the result. Since $(x^2+ax+b)^2 = x^4+2ax^3+(a^2+2b)x^2+2bax+b^2$, the given expression matches the square of a trinomial best if $a = 4$ and $b = 2$ (we get this result by comparing coefficients).

Then we can write the given condition as

$$(x^2+4x+2)^2 - 4(x-2) = y^2.$$

(this process is similar to completing the square in solving a quadratic equation). From this form of the problem, we see immediately that $x = 2$ will give the expression the value $14^2 = 196$.

To explore the possiblity of other solutions, we can rewrite the given expression as

$$(x^2+4x+2)^2 - y^2 = 4(x-2).$$

Let $u = x+2$, so that $y^2 = (u^2-2)^2 - 4(u-4)$. We can then distinguish three cases:

(i) Suppose $u > 4$. Then we can show that $(u-3)^2 < y^2 < (u-2)^2$. Then, since y is included between the squares of two consecutive integers, it cannot itself be a perfect square.

Since $u > 4$ by assumption, $4(u-4) > 0$, and $y^2 = (u^2-2)^2 - 4(u-4) < (u^2-2)^2$. Next we calculate the difference $y^2 - (u^2-3)^2$, and show that it is always positive. This will prove that $y^2 > (u-3)^2$. We have:

$$y^2 - (u^2-3)^2 = (u^2-2)^2 - 4(u-4) - (u^2-3)^2 = 2u^2 - 5 - 4u + 16$$
$$= 2u^2 - 4u + 11 = 2(u-1)^2 + 9 > 0.$$

(ii) Suppose $u < -4$. Then, writing $v = -u$, we will again show that y^2 is contained between the squares of consecutive integers:

(a) $y^2 = (v^2-2)^2 + 4(v+4)$, so $y^2 > (v^2-2)^2$.

(b) $(v^2-1)^2 - y^2 = 2(v-1)^2 - 21$, so for $v >= 5$, this difference is positive, and $y^2 < (v^2-1)^2$.

(iii) For $u = -4, -3, -2, -1, 0, 1, 2$ or 3, a direct calculation shows that no new perfect squares are produced.

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE
Spring, 1987
Senior A Division Contest Number Five

SOLUTIONS

S87S25. There are six sides to be toasted, and each side takes 30 seconds, so the total time cannot be less than 90 seconds. This minimum can be achieved, for instance, as follows:

first 30 seconds: front of slice A and front of slice B
middle 30 seconds: back of slice A and front of slice C
last 30 seconds: back of slice B and back of slice C.

S87S26. Let $x = \sqrt{1986}$, $y = \sqrt{1987}$. Then $a = x+y$, $b = 2\sqrt{xy}$, and $c = 2\sqrt{(x^2+y^2)/2}$, so

$$a^2 = x^2+y^2+2xy,$$
$$b^2 = 4xy,$$
$$c^2 = 2(x^2+y^2).$$

Hence $c^2-a^2 = x^2+y^2-2xy = (x-y)^2 > 0$, and $c > a$.
Also, $a^2-b^2 = x^2+y^2-2xy = (x-y)^2 > 0$, so $a > b$.

S87S27. There are $\binom{15}{5}$ ways to choose a committee of five: this number is $7 \cdot 13 \cdot 3 \cdot 11$. There are $\binom{5}{2} = 10$ ways to choose the girls, and $\binom{10}{3} = 5 \cdot 3 \cdot 8$ ways to choose the boys. The probability is thus

$$\frac{5 \cdot 3 \cdot 8 \cdot 10}{7 \cdot 13 \cdot 3 \cdot 11} = \frac{400}{1001}.$$

S87S28. Let the radius of the sphere be r, and note first that if A and B are natural numbers, then so is r. Now

$1000 \leq 4r^2 < 10000$, so $250 \leq r^2 \leq 2500$, and $15 < r \leq 50$.
Also, $1000 \leq 4r^3/3 < 10000$, so $750 \leq r^3 \leq 7500$, and $9<r<20$.
Hence $15 < r < 20$. Since r must be a multiple of 3, $r = 18$.

S87S29. Clearly, the sum is greater than 1. But is it as much as 2? The answer is no.

Note that
$$1/1^2 < 1/1 \cdot 2$$
$$1/2^2 < 1/2 \cdot 3$$
$$1/3^2 < 1/3 \cdot 4$$
$$\vdots$$
$$1/n^2 < 1/n \cdot (n-1).$$

Hence the required sum is less than

$$S = 1 + 1/1 \cdot 2 + 1/2 \cdot 3 + 1/3 \cdot 4 + \ldots + 1/49 \cdot 50$$
$$= 1 + (1/1 - 1/2) + (1/2 - 1/3) + (1/3 - 1/4) + \ldots + (1/50 - 1/49) =$$
$$= 1 + 1/1 - 1/50 = 2 - 1/50,$$ and the next higher integer is indeed 2.

The "telescoping sum" S can easily be guessed by computing the first few terms and noting the pattern.

S87530. The key to this problem is to take different planar cross sections of the figure.

Let the centers of the three equal circles be O_1, O_2, O_3, let the center of the fourth circle be O_4, and let the center of the sphere be O. Note that O is equidistant from any point on any of the circles, so OO_i is perpendicular to any radius of circle O_i, for i = 1,2,3,4.

Clearly, triangle $O_1O_2O_3$ is equilateral. Let us find its side by examining a cross section along the plane of points O, O_1, O_2. If M is the point of tangency of circles O_1,O_2 then $MO_1 = 10 = (1/2)OM$, so triangle MOO_1 is a 30-60-90 triangle. Hence $OO_1 = 10\sqrt{3}$. Also, triangle O_1O_2O is isosceles, and contains a 60° angle, so it is equilateral, and $O_1O_2 = OO_1 = 10\sqrt{3}$.

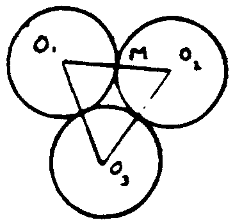


fig i

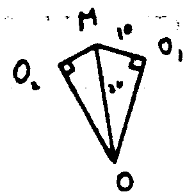


fig ii

Let T be the center of triangle $O_1O_2O_3$: by symmetry, it must lie on line OO_4. Looking along the plane of this triangle (figure iii), we find that $O_1T = 10$.

Next we look at a cross-section through the plane of points O, O_1, O_4 (figure iv). Let K be the point of tangency of circles O_1, O_4, and draw KN perpendicular to O_1T. Since angles KO_1T, TOO_1 are both complementary to angle TO_1O, $\angle KO_1T = \angle TOO_1$, so triangles O_1KN, OO_1T are similar. Using the Pythagorean theorem in triangle OTO_1, we find $OT^2 = OO_1^2 - O_1T^2 = 200$, so $OT = 10\sqrt{2}$. Then $O_1K:O_1N = OO_1:OT$, so $10:O_1N = 10\sqrt{3}:10\sqrt{2}$, and $O_1N = 10\sqrt{2/3}$. Hence $O_4K = NT = O_1T - O_1N = 10 - 10\sqrt{273}$, and this is the required radius.

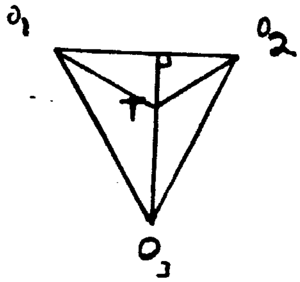


fig iii

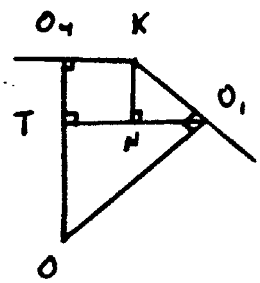


fig iv