New York City Interscholastic Mathematics League

JUNIOR DIVISION

Part I Time: 11 Minutes Contest Number One Spring, 1987

S87J1. What is the base of the system of numeration in which there are 11 days in a week? Write your answer in decimal notation.

S87J2. In triangle ABC, $AC \perp BC$, AC = 10, and BC = 24. Compute the length of the median to side AB.

Part II Time: 11 Minutes NYCIML Contest Number One Spring, 1987

S87J3. In a class of 24 students, each student has, as a pet, either a cat or a dog, or both. Of the students who have cats as pets, 1/3 also have dogs. Of the students who have dogs as pets, 1/4 also have cats. How many students have cats?

S87J4. If a is an irrational number and r is a rational number, which one of the following expressions, when representing a real number, could be rational?

(i) $\sqrt{a+\sqrt{r}}$ (ii) $\sqrt{\sqrt{a}+\sqrt{r}}$ (iii) $\sqrt{r}+\sqrt{a}$ (iv) $\sqrt{a+r}$

Part III Time: 11 Minutes NYCIML Contest Number One Spring, 1987

S87J5. The number $1920 = 2^{11} - 2^{7}$ is the difference of two positive integral powers of 2. Find the next largest such number.

S87J6. In triangle ABC, AB = 10, BC = 12 and AC = 20. Three circles have their centers at points A, B, and C. If these three circles are externally tangent to each other in pairs, find the radius of the largest of the circles.

ANSWERS

1. 6
2. 13
3. 12
4. $\sqrt{a+\sqrt{r}}$ [or choice (i)]
5. 1984
6. 11

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Part I   Time: 11 Minutes   Contest Number Two   Spring, 1987

S87J7. In triangle ABC, AB = BC. Point T is on BC, and BT = TA = AC. Compute the degree measure of angle ABC.

S87J8. During one year, an oil well increased its daily rate of production twice, both times by the same percentage. If it started the year producing oil at a rate of 600 barrels per day, and ended by producing 726 barrels per day, compute the percent of increase each time.

Part II   Time: 11 Minutes   NYCIML Contest Number Two   Spring, 1987

S87J9. Mike and Pete opened a luncheon restaurant. In this restaurant, a glass of milk, three hotdogs and 7 Munchie bars cost \$14, while a glass of milk, four hotdogs, and 10 Munchie bars cost \$17. Find the cost, in dollars, of two glasses of milk, three hotdogs, and five Munchie bars.

S87J10. Two circles with radii 6 and 10 are externally tangent. Compute the length of their common external tangent.

Part III   Time: 11 Minutes   NYCIML Contest Number Two   Spring, 1987

S87J11. If $\log_{16} 2 = m$, express in terms of m the value of $\log_4 5$.

S87J12. The greatest common divisor of two natural numbers is 24, and the least common multiple is 2496. What is the smallest <u>sum</u> these two numbers can have?

ANSWERS

7. 36 or 36°   9. 19 or 19   11. 1/2m - 1/2 or equivalent

8. 10 or 10%   10. $4\sqrt{15}$ or equivalent.   12. 504

New York City
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Part I Time: 11 Minutes Contest Number Three Spring, 1987

S87J13. In circle O, radii OA and OB are perpendicular. Point P is chosen on minor arc AB, and points X, Y on OA, OB respectively so that $PX \perp OA$ and $PY \perp OB$. If OA = 15, compute the length of XY.

S87J14. The quadratic equation $x^2 + qx + q = 0$, where q is a real number, has one root equal to -3. Compute the numerical value of the other root of the equation.

Part II Time: 11 Minutes NYCIML Contest Number Three Spring, 1987

S87J15. Find a perfect square whose decimal representation consists of four digits, of which the first two are identical and the last two are also identical.

S87J16. Romeo has lost the last digit of Juliet's telephone number. He tries to call her by picking the last digit at random and dialing the number. What is the probability that Romeo will succeed after only one or two tries (assuming Juliet's line is never busy)?

Part III Time: 11 Minutes NYCIML Contest Number Three Spring, 1987

S87J17. The unique real solution to the equation

$$x\sqrt[3]{x^2} + 16 = 8\sqrt[6]{x^5}$$

can be expressed as 2^a. Compute the rational number a.

S87J18. Three sides of a triangle are 10, 12, and 20. Three spheres are all tangent to the plane of the triangle, one at each vertex, and the spheres are externally tangent to each other in pairs. Find the radius of the largest of the three spheres.

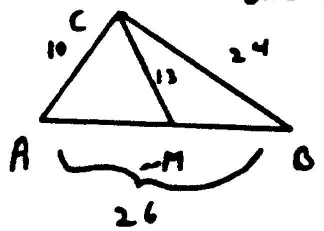
ANSWERS

13. 15	15. 7744	17. 12/5
14. -3/2	16. 0.19 or equivalent	18. 12

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE
Spring, 1987
Junior Division Contest Number One

S87J1. If the base is b, then b + 1 = 7 (using decimal notation throughout), so b = 6.

S87J2. The median to the hypotenuse of a right triangle is 1/2 the hypotenuse. Hence CM = 13.



S87J3. If the number of students with both pets is b, then there are 2b students with cats only, and 3b students with dogs only. Hence b + 2b + 3b = 24 (since each student falls into exactly one category), and b = 4. The number of students who have cats is 3b, or 12.

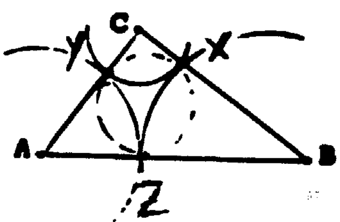
S87J4. If a is irrational, then the numbers ra (if it is real), 2a, and a+r must also be irrational. Also, the sum of a rational and an irrational is an irrational. Hence $a+r$ is irrational so $\sqrt{a+r}$ is irrational
$r+\sqrt{a}$ is irrational so $\sqrt{r+\sqrt{a}}$ is irrational
$\sqrt{a}+\sqrt{r}$ is irrational so $\sqrt{\sqrt{a}+\sqrt{r}}$ is irrational.
But if $a = -\sqrt{r}$, then $\sqrt{a+\sqrt{r}} = 0$, which is a rational number.

S87J5. We know that $2^{11} - 2^7 = 1920$. Let $N = 2^a - 2^b$. Clearly, if $a < 11$ and $b > 7$, then $N < 1920$. If $a = 11$, and $b = 6$, $2^{11} - 2^6 = 2^6(2^5-1) = 2^6 \cdot 31 = 1984$. If $a = 11$ and $b < 6$, then N is larger than this number. If $a > 11$ and $b > 7$, N can only get larger.

S87J6. Since the common centerline of two tangent circles passes through the point of tangency, the three circles meet at points on the sides of the triangles. Call these points X, Y, and Z (see diagram). Then, letting AY = AZ = y, BX = BZ = x, CX = CY = z, we have the equations:

$$x + y = 10$$
$$y + z = 20$$
$$x + z = 12$$

We can solve these simultaneously by adding: $2x + 2y + 2z = 42$, and $x+y+z = 21$. Subtracting each of the original equations from this relationship gives $z = 11$, $y = 9$, $x = 1$, so the largest radius is 11.



Note that X, Y, and Z are the points of contact of the circle inscribed in triangle ABC.

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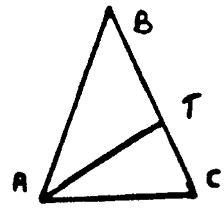


S87J7. Triangles CAT, BAT are isosceles, so <ATC = <C = <CAB. If <ACT = x, then <B = <TAC = 180 - 2x, and <BAC = 2(<TAC), so x = 2<B = 360 - 4x, or 5x = 360, and x = 72. Hence <ABC = 36°.

S87J8. Suppose the increase is y%. Then let x = y + 1, and $600x^2 = 726$. Then $x^2 = 726/600 = 121/100$, and x = 1.1, so y = .1 = 10%.

S87J9. If a glass of milk costs a dollars, a hotdog b dollars, and a Munchie bar c dollars, then a+3b+7c = 14, and a+4b+10c = 17. We need to find the value of 2a+3b+5c. We have:

	a+4b+10c = 17	(P)
	a+3b+7c = 14	(Q)
Subtracting:	b+3c = 3	(R=P-Q)
Adding:	2a+7b+17c = 31	(S=P+Q)
	4b+12c = 12	(4R=4P-4Q)
Subtracting:	2a+3b+5c = 19	(S-4R = -3P+5Q)

This process is the same as that of taking linear combinations of vectors, as shown in the column at the far right.

S87J10. If the points of tangency are X and Y, and the centers are O and P (see figure), then we can drop perpendicular OZ to YP, and look at triangle OPZ. Here, OP=16, OZ=XY, and ZP=YP-YZ=YP-XO=10-6=4. Then, using the Pythagorean theorem,

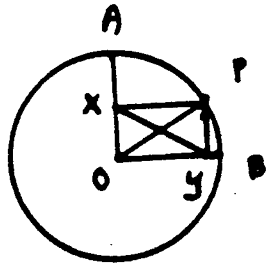
$XY^2 + 4^2 = 16^2$, or $XY^2 = 256 - 16 = 240$, and $XY = \sqrt{240} = 4\sqrt{15}$

S87J11. Since $(\log_a b)(\log_b a) = 1$, $\log_2 10 = 1/m$.
Since $(\log_a b)(\log_b c) = \log_a c$, $\log_4 10 = \log_4 2 \cdot \log_2 10 = (1/2)(1/m) = 1/2m$.
Since $\log_4 10 = \log_4 5 \cdot 2 = \log_4 5 + \log_4 2$, $1/2m = \log_4 5 + 1/2$, and $\log_4 5 = 1/2m - 1/2$.

S87J12. If the numbers are a and b, then let a = 24x, b = 24y. The product of two natural numbers is equal to the product of their GCD and their LCM. Hence $ab = 24x^2y = 24 \cdot 2496$, so xy = 104 = 8·13. Since the product ab is constant, the sum of the two numbers is largest when they are nearly equal, which means that x and y should be as close as possible. Hence we may take x = 8, y = 13. Then a = 192, b = 312, and a+b = 504.

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE
Spring, 1987
Junior Division Contest Number Three

S87J13. Since quadrilateral XPBO is a rectangle, diagonals XY, OP are equal, and XY = OP = OA = 15.



S87J14. Since -3 is a root of the equation, $(-3)^2 - 3q + q = 0$, so $q = 9/2$. Then the product of the roots is 9/2, so the second root is $9/2 \div (-3) = -9/6 = -3/2$.

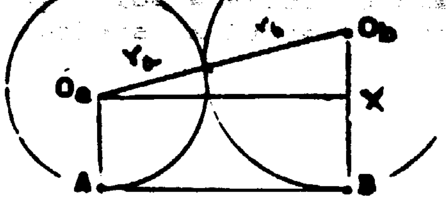
S87J15. If the required number is N, then 11 must divide N. If a prime divides a perfect square, the square of the prime must also divide the square, so $N = 121 \cdot x^2$ for some integer x. Since $N < 10000 = 100^2$, x must be less than 10. Trying x = 9, 8, etc., we find that x = 8 is the only solution, and $N = 121 \cdot 64 = 7744$.

To see that N is divisible by 11, note that the "alternating sum" of its digits is zero.

S87J16. The probability of Romeo missing on any given try is 0.9, so the probability that he will miss on two consecutive tries is $.9^2 = 0.81$, and the probability that he will succeed is 1 - 0.81 = 0.19.

S87J17. We can write the given equation as $x^{5/3} + 16 = 8x^{5/6}$. Letting $y = x^{5/6}$, we find $y^2 - 8y + 16 = 0$, and y = 4. Then $x^{5/6} = 4$, so $x = 4^{6/5} = 2^{12/5}$.

S87J18. We will solve the general problem first. Let the triangle be ABC, with sides of length a, b, and c, and suppose O_a, O_b are the centers of the spheres tangent at points A and B respectively, with r_a, r_b the respective radii of the spheres. Without loss of generality, we assume that $r_b > r_a$. We first look along the plane of points A, B, O_a and O_b.



We know the length of a common tangent to two tangent circles, and we wish to relate their radii (compare problem S87J10.). Drawing $O_aX \perp$ to O_bB, we find:

$O_aX = AB = c$,
$O_bX = O_bB - BX = r_b - r_a$
$O_aO_b = r_a + r_b$.

Then, using the Pythagorean theorem in triangle O_aO_bX, we find:

$(r_a + r_b)^2 = c^2 + (r_b - r_a)^2$, or, after expanding and simplifying,

$4r_ar_b = c^2$, and $r_ar_b = c^2/4$.

Although we have assumed that $r_b > r_a$, the above relationship is symmetric in these quantities, and so does not really depend on the assumption. Since the situation is symmetric also in a, b, and c, we can easily derive analegous relations: $r_ar_c = b^2/4$
$r_br_c = a^2/4$.
These three equations can be solved for r_a, r_b, r_c by multiplying them together: $r_a^2r_b^2r_c^2 = a^2b^2c^2/64$, so $r_ar_br_c = abc/8$. Dividing this equation by each of the three original equations gives:

$r_a = bc/2a$, $r_b = ac/2b$, $r_c = ab/2c$.

In the present situation, the values for the radii are 3, 12, and 25/3, of which 12 is the largest.