Fall 1986 – Senior B – 1 – Questions

F86B1

The fraction a/b (where a and b are integer) is in lowest terms. If the denominator is added to both the numerator and the denominator, the fraction is doubled. Find the fraction a/b.

F86B2

The minute hand of a clock travels $k\pi$ radians between 1:00 P.M. and 3:15 P.M. that same afternoon. Compute k.

F86B3

On a rectangular coordinate system, the line that passes through (2,-3) and (1,4) also passes through the point (11,k). Compute k.

F86B4

Find all real numbers x such that $\log\left[(x-1)^2\right] = \log(x-1)$, where the base of the logarithms is 10.

F86B5

In square ABCD, side AB = 12. Point E is on AD and point F is on CD such that AE:ED = CF:FD = 2:1. Find the length of the perpendicular from point BN to line EF.

F86B6

Find all real values of a for which the equations $x + 2ay = 1$ and $8ax + y = 5$ have <u>no</u> common solutions.

Answers

1. 1/3
2. 9/2
3. –66
4. 2
5. $10\sqrt{2}$
6. 1/4, -1/4; both required

Fall 1986 – Senior B – 2 – Questions

F86B7

Find all real numbers x such that $(x+3)^2 = x+3$.

F86B8

The degree-measures of the angles of triangle I are a, b, and c. The degree-measure of the angles of triangle II are d, e, and f. If d = 2a + 10 and e = 2b, write an equation expressing f in terms of c only.

F86B9

A natural number less than 24 is chosen at random. Compute the probability that this number is relatively prime to 24 (note that 1, the smallest natural number, is relatively prime to any number).

F86B10

In the right triangle ABC, AC is perpendicular to BC, AC = 5, and AB – BC = 3. Compute the value of AB + BC.

F86B11

Find all ordered pairs (x,y) of real numbers which satisfy $\frac{x+y}{xy} = \frac{3}{10}$ and

$\frac{3x-4y}{5xy} = \frac{-13}{25}$.

F86B12

A circus tent is held up by two unequal vertical poles, 20 and 15 feet in height. A wire leads from the top of each pole to the base of the other pole. How high, in feet, above the ground is the point where the two wires cross?

Answers

7. -3, -2: both required
8. f = 2c – 190: equation required
9. 8/23
10. 25/3
11. (2,-5): ordered pair required
12. 60/7

Fall 1986 – Senior B – 3 – Questions

F86B13

When old uncle Nicholas died, he left a sum of money to his nephews, to be given to them over the years. Every year, each nephew got a sum equal to a constant times his age that year. If nephew Willie got $250 the first year and $260 the third year, how old was Willie when his uncle died?

F86B14

In quadrilateral PQRS, diagonals PR and QS are perpendicular. If PR = 10 and QS = 14, compute the area of quadrilateral PQRS.

F86B15

Find all ordered pairs (x,y) of real numbers such that x + y = 6 and xy = 5.

F86B16

Two circles of radius 1 and 7 are coplanar, and their centers are 10 units apart. Find the length of their common external tangent.

F86B17

Find all ordered triples (a,b,c) of real numbers such that

$$\frac{ab}{a+b}=\frac{1}{3}$$
$$\frac{ac}{a+c}=\frac{1}{4}$$
$$\frac{bc}{b+c}=\frac{1}{5}$$

F86B18

The expression $\left(4+4i\sqrt{3}\right)^{18}$ (where $i^2=-1$) represents a real number. This same real number can be written as 2^x. Compute the real number x.

Answers

13. 49 or 50
14. 70
15. (1,5) (5,1): both ordered pairs required
16. 8
17. (1, 1/2, 1/3) ordered triple required
7. 54

Fall 1986 – Senior B – 4 – Questions

F86B19

If all numerals are written in base ten notation, how many digits are in the numeral representing 61224^2?

F86B20

The perimeter of a rhombus is 80 centimeters, and the ratio of its two diagonals is 3:4. Compute the area of the rhombus.

F86B21

Compute the value of the expression $(1+2x)(1-2x)(1+4x^2)$ when x = 3.

F86B22

The perimeter of triangle $A_1B_1C_1$ is 30. Triangle $A_2B_2C_2$ is formed by connecting the midpoints of the sides of triangle $A_1B_1C_1$. Triangle $A_3B_3C_3$ is formed by connecting the midpoints of the sides of triangle $A_2B_2C_2$. This process is continued to form an infinite series of triangles. Compute the infinite sum of all the perimeters of these triangles.

F86B23

Find all ordered pairs (x,y) of rational numbers such that $(3i-8)x+(2i-5)y=i$, where $i^2=-1$.

F86B24

If a and b are the roots of the equation $x^2-8x+3=0$, compute the numerical value of $\dfrac{7a^2-5ab+7b^2}{a^2+2ab+b^2}$.

Answers

19. 10
20. 384
21. –1295
22. 60
23. (-5,8): ordered pair required
24. 391/64

Fall 1986 – Senior B – 5 – Questions

F86B25

In parallelogram ABCD, point E is on side AB, and AE = (1/3)AB. If ED intersects AC at F, compute the ratio DE:DF.

F86B26

A restaurant has a full jar of ketchup. On Monday, 1/6 of the ketchup was used. On Tuesday, three times as much ketchup was used as was used on Monday. On Wednesday, ten ounces of ketchup was used, and the jar was emptied. How many ounces of ketchup did the jar hold originally?

F86B27

Find all ordered pairs (x,y) of real numbers such that $x + y = 12$, and $x^2 + y^2 = 90$.

F86B28

In triangle ABC, $m\angle C = 90$. Semicircles are drawn <u>outside</u> the triangle, with diameters AB, BC and CA. If the area of the semicircle on AB is 200π, and the area of the semicircle on BC is 128π, the area of the semicircle on AC can be represented as $k\pi$. Compute the rational number k.

F86B29

The complex number $\frac{2-3i}{1+i}$ can be represented as a + bi. Find the ordered pair of rational numbers (a,b).

F86B30

In trapezoid ABCD, base AB = 5 and base CD = 9. Points X and Y are on AD and BC respectively, and $XY \parallel AB$. If the areas of trapezoids ABYX and XYCD are equal, compute the length of line segment XY.

Answers

25. 4:3 or equivalent
26. 30 or 30 oz.
27. (9,3), (3,9)
    both ordered pairs required
28. 72
29. (-1/2, -5/2): ordered pair required
30. $\sqrt{53}$ or equivalent

Fall 1986 – Senior B – 1 – Solutions

F86B1

We have: $\frac{a+b}{b+b} = \frac{a+b}{2b} = \frac{2a}{b}$. Since b ≠ 0, we can divide by b and cross-multiply, to get a + b = 4a, and b = 3a. The fraction is 1/3.

F86B2

From 1:00 to 3:00, the minute hand travels 2π radians. From 3:00 to 3:15 it travels … radians. Altogether, it travels $2\pi + \pi / 2 = 9\pi / 2$, so $k = 9/2$.

F86B3

The slope of the line is $\Delta y / \Delta x = -7$, so:

$$-7 = (k-4)/(11-1) = (k-4)/10$$

$$k - 4 = -70$$

$$k = -66$$

F86B4

Let y = x – 1. Then $\log y^2 = 2\log y = \log y$, so $\log y = 0$. Then y = 1, so x – 1 = 1, and x = 2.

F86B5

Since DF = (1/3)DC = 4, and triangle DEF is a right isosceles triangle, $EF = 4\sqrt{2}$. Then $DX = (1/2)EF = 2\sqrt{2}$, since the median to the hypotenuse of a right triangle is half the hypotenuse. Then triangle BCD is also an isosceles right triangle, so $BD = 12\sqrt{2}$, and $BC = BD - BX = 12\sqrt{2} - 2\sqrt{2} = 10\sqrt{2}$.

F86B6

For simultaneous equations to have no solution in common, it is necessary that their coefficients be proportional. Hence: $1/8a = 2a/1$, or $16a^2 = 1$, and $a = \pm 1/2$. But this condition is not sufficient: the two equations might have infinitely many common solutions. But then, if a = ¼, the equations become:

$$2x + y = 2$$

$$2x + y = 5$$

which clearly have no common solution. The value a = -1/4 can be checked similarly.

Fall 1986 – Senior B – 2 – Solutions

F86B7

One way to solve this problem is to put the quadratic equation in standard form and solve. Or, we could let y = x + 3. Then $y^2 = y$, so $y^2 - y = 0$, or y(y – 1) = 0, and y = 0, 1. Then x = y – 3 = -3 or –2.

F86B8

We have d + e + f = 180, or 2a + 10 + 2b + f = 180, so 2a + 2b + f = 170. Since a + b + c = 180, we know that 2a + 2b = 360 – 2c. Then 360 – 2c + r = 170, and f = 2c – 190.

F86B9

Since the divisors of 24 include all the primes less than $\sqrt{24}$, the only numbers relatively prime to 24 in the given range are 1 and the primes themselves: 5, 7, 11, 13, 17, 19, and 23. There are eight of these, and 23 numbers to choose from altogether, so the probability is 8/23.

F86B10

If AB = c, AC = b, and BC = a, then $b^2 = c^2 - a^2 = (c+a)(c-a)$. If AB + AC = x, we then have 25 = 3x, so x = 25/3

F86B11

The given equations are equivalent to

$$1/y + 1/x = 3/10$$

$$3/5y + 4/5x = -13/25$$

Let a = 1/y, b = 1/x. Then a + b = 3/10 and 3a – 4b = -13/5 = -26/10. Solving simultaneously for a and b, we find a = -1/5, b = ½. Then x = 2, y = -5.

F86B12

Let the poles be AC and BD (see diagram) and suppose TU represents the height of the point of intersection. From similar triangles BTU, BAC, we have x/15 = q/(p + q). Adding, we find x/15 + x/20 = (p + q)/(p + q) = 1, so 35x/300 = 1, and x = 300/35 = 60/7.

Note that the answer does not depend on the value of p + q: the answer is the same no matter how far apart the two poles are.

Fall 1986 – Senior B – 3 – Solutions

F86B13

Since Willie received $5 more each year, he must have been 250/5 = 50 years old when his uncle died.

F86B14

Using absolute value for area, we have:

$$|PSQ| = (1/2)PT \cdot SQ$$
$$|RSQ| = (1/2)TR \cdot SQ$$
$$\begin{aligned} |PQRS| &= |PSQ| + |RSQ| \\ &= (1/2)SQ \cdot (PT + TR) \\ &= (1/2)PR \cdot SQ \\ &= (1/2)10 \cdot 14 \\ &= 70 \end{aligned}$$

In general, the area of a quadrilateral whose diagonals are perpendicular is half the product of its diagonals.

F86B15

The easiest way to solve this sort of system is to notice that x and y must be the roots of the new equation $z^2 - 6z + 5 = 0$. The roots of this equation are 1 and 5. Since the given system is symmetric in x and y, either of these roots can be the value of x, and the other root is the corresponding value of y.

F86B16

If we draw O_1V perpendicular to O_2U (see diagram), then $TU = O_1V$, and $UV = O_1T$, so $O_2V = O_2U - O_2V = 7 - 1 = 6$. Using the Pythagorean theorem in triangle O_1O_2V, we find $O_1V^2 = O_1O_2^2 - O_2V^2 = 100 - 36 = 64$. Thus $TU = O_1V = 8$.

F86B17

The given system is equivalent to:

(i) $1/b + 1/a = 3$
(ii) $1/c + 1/a = 4$
(iii) $1/c + 1/b = 5$.

Adding, we find $2/a + 2/b + 2/c = 12$, so $1/a + 1/b + 1/c = 6$. Subtracting each of the equations (I), (ii), (iii) from this relationship, we find $1/a = 1$, $1/b = 2$, $1/c = 3$, so that (a,b,c) = (1, 1/2, 1/3).

F86B18

It is easiest to compute powers of complex numbers if they are put in polar form, and this can be done by recognizing the 30-60-90 triangle in the diagram. The given number is $8(\cos\pi/3 + i\sin\pi/3) = 8cis\pi/3$, and De Moivre's theorem shows that

$(8cis\pi/3)^{18} = 8^{18}cis18\pi/3 = 8^{18}cis6\pi = 8^{18} = (2^3)^{18} = 2^{54}$

Fall 1986 – Senior B – 4 – Solutions

F86B19

Since $10^4 < 61224 < 10^5$, we have $10^8 < 61224^2 < 10^{10}$, and 61224 has 9 or 10 digits. Now $\sqrt{10^9} = 10^4\sqrt{10} < 10^4 \cdot 4$, while $61224 > 6 \cdot 10^4 > 4 \cdot 10^4 > \sqrt{10^9}$. Hence $61224^2 > 10^9$ and has ten digits.

F86B20

If the diagonals are 3x and 4x, then (see diagram) $(3x)^2 + (4x)^2 = 20^2$, so x = 4. Then each small triangle has sides 12, 16, and 20, and the diagonals of the rhombus are 24 and 32. Its area is half the product of its diagonals, or 384.

F86B21

Simplifying the given expression, we have: $(1-4x^2)(1+4x^2) = (1-16x^4)$. Substituting x = 3, we find that $1 - 16 \cdot 81 = -1295$.

F86B22

Since the perimeter of each triangle is half as big as that of the previous triangle, the perimeters form an infinite geometric progression, whose first term is 30 and whose common ratio is ½. The sum to infinity of this progression is 30/(1 – 1/2) = 60.

F86B23

Eliminating parenthesis, and separating real and imaginary parts gives:
$3xi - 8x + 2yi - 5y = -8x - y + (3x - 2y)i = 0 + 1i$. Equating real and imaginary parts gives

$$\begin{aligned} -8 - 5y &= 0 \\ 3x + 2y &= 1 \end{aligned}.$$

Solving simultaneously shows $x = -5$, $y = 8$.

F86B24

The given expression can be written as: $\dfrac{7(a^2+b^2)-5ab}{(a+b)^2}$.

The product ab of the roots of the given equation is 3, and the sum a + b = 8, so the value of the expression is: $\dfrac{7(a^2+b^2)-15}{64}$. Now $a^2 + b^2 = (a+b)^2 - 2ab = 64 - 6 = 58$, so the given expression equals $(7 \cdot 58 - 15)/64 = 391/64$.

Fall 1986 – Senior B – 5 – Solutions

F86B25

From similar triangles AEF, DFC, we find DF:FE = DC:AE = 3:1, so DE:DF = 4:3.

F86B26

On Tuesday, 3/6 of the jar was used, so the amount consumed on Monday and Tuesday is 4/6 of the jar. Thus 1/3 of the jar was left on Wednesday, and this was 10 ounces. Hence the jar originally contained 30 ounces.

F86B27

Squaring the first equation produces $x^2 + y^2 + 2xy = 144$. Substituting the value from the second equation gives 90 + 2xy = 144, so 2xy = 54, and xy = 27. Now we are in the situation of problem F86B15: x and y are the roots of the equation $z^2 - 12z + 27 = 0$. These roots are 9 and 3, so we can have (x,y) = (9,3) or (x,y) = (3,9).

F86B28

The area of a semicircle of diameter, d, is $\pi d^2/8$. Then $AB^2/8 = 200$, so AB = 40. Also, $BC^2/8 = 128$, so BC = 32. The Pythagorean theorem then shows that AC = 24 and the semicircle on AC is 72π.
Note that the sum of the areas of the two smaller semicircles equals the area of the semicircle on the hypotenuse. This generalization of the Pythagorean theorem is true for any three similar figures constructed on the sides of a right triangle.

F86B29

We have: $\frac{(2-3i)(1-i)}{(1+i)(1-i)} = \frac{2-3i-2i-3}{1+1} = \frac{-1-5i}{2}$, so $(a,b) = (-1/2, -5/2)$.

F86B30

Let XY = x. Extend DA and CB to intersect at P. Then triangle PAB, PXY, PDC are similar, and the ratio of their sides is 5:x:9. Hence the ratio of their areas is $25 : x^2 : 81$. Using absolute value for area, we then have $|PAB| = 25k$, $|PXY| = x^2 k$, $|PCD| = 81k$ (for some real number k), and $|PCD| - |PXY| = |PXY| - |PAB|$, or $81k - x^2 k = x^2 k - 25k$. This equation leads to $2x^2 = 81 + 25 = 106$, and $x = \sqrt{53}$.

January 26, 1987

Dear Math Team Coach,

Enclosed is your copy of the FALL, 1986 NYCIML contests that you requested on the application form.

The following are acceptable alternative answers for the enclosed contests:

	Question	Correct answer
Senior A	F86S23	π or 180°
Senior B	F86B13	49 or 50

Have a great spring term.

Sincerely yours,

Richard Geller

Secretary, NYCIML

~~ONLY~~ ~~complete one~~