## Fall 1986 – Senior A – 1 – Questions

### *F86S1*

Find a divisor of the number 496 which is between 50 and 100.

### *F86S2*

By “center” of a face of a cube we will here mean the intersection of the diagonals of that face. Id every pair of these centers of the faces of a cube are connected by line segments, how many such line segments will there be?

### *F86S3*

If the sum of two numbers is 6, and the sum of their squares is 10, compute the product of the two numbers.

### *F86S4*

In the triangle ABC, median AM = 4 and median BN = 3. If $AM \perp BN$, compute the area of triangle ABC.

### *F86S5*

The natural number N is written in decimal notation. If the digit 2 is appended to the right, the resulting number is a multiple of 9. When this new number is divided by 9, the quotient is 21 more than N. Find N.

### *F86S6*

If M = 1/6!, and N = $N = 1 \cdot 1! + 2 \cdot 2! + 3 \cdot 3 + 4 \cdot 4 + 5 \cdot 5 + 6 \cdot 6 + 7 \cdot 7 + 8 \cdot 8 + 9 \cdot 9 + 1$, compute the product MN.

### *Answers*

1. 62
2. 15
3. 13
4. 8
5. 187
6. 5040

# Fall 1986 – Senior A – 2 – Questions

### *F86S7*

Richard traveled by train from New York to Chicago. He fell asleep half way through the journey, then woke up in Ohio. After he awoke, he traveled half as far as he had traveled while he was asleep, then arrived in Chicago. For what fraction of the total journey was he asleep?

### *F86S8*

In triangle ABC, $AC \perp BC$, AC = x, BC = y, and the altitude from C to AB is 6. Compute $\frac{1}{x^2} + \frac{1}{y^2}$.

### *F86S9*

A car leaves New York City for Albany at noon, travelling at a constant rate of 45 miles per hour. A second car leaves New York City 20 minutes later and travels along the same route at a constant rate of 55 miles per hour. How far from New York City will the second car overtake the first?

### *F86S10*

Compute the real number represented by $\sqrt[3]{20+14\sqrt{2}} \cdot \sqrt[3]{20-14\sqrt{2}}$.

### *F86S11*

Every two digit multiple of 11 (in decimal notation) is divisible by each of its two (identical) digits. How many other two digit decimal numbers are divisible by each of its two digits?

### *F86S12*

Polyhedron PABCD is a pyramid in which ABCD is a square. Diagonals AC and BC intersect at X, and PX is perpendicular to the plane of ABCD. The bisector of angle PDA (in plane PDA) intersects PA at M, and plane CDM intersects PB at N. If AB = 5, AP = 10, compute MN.

### *Answers*

7. 1/3
8. 1/36
9. 82.5 or 82.5 miles
10. 4
11. 5
12. 10/3

## Fall 1986 – Senior A – 3 – Questions

### *F86S13*

Twenty cars are parked in a garage, all either red, white, or blue, and at least one car of each color. If there are nine times as many red cars as whiter cars, how many blue cars are there?

### *F86S14*

The sum of the first three terms of an arithmetic progression is 66, and the product of the second and third terms is 528. Compute the sum of the first 100 terms.

### *F86S15*

If $a = \frac{133}{221}$ and $b = \frac{221}{399}$, compute $\left(a+b\right)^2 - \left(a-b\right)^2$.

### *F86S16*

It is well known that an angle of $20°$ is not constructible with straightedge and compass, while angles of $18°$ and $30°$ are (for instance, by bisecting the exterior angle of a regular pentagon or equilateral triangle, respectively). Find the degree-measure of the smallest angle whose degree-measure is a natural number, and which is constructible with straightedge and compass.

### *F86S17*

A railroad terminal is served by two commuter lines. Every 16 minutes, a train from line A arrives, and a crowd of commuters gets off and exits the station. Every 21 minutes, a train from line B arrives, and a crowd of commuters gets off and exits the station. An observer watching the exit counts the trains by noting how many crowds of commuters exit, but she cannot distinguish between two crowds of commuters if the trains arrive 6 minutes or less apart. How many crowds does the observer see during a three hour period, starting with the arrival of two trains simultaneously?

### *F86S18*

In tetrahedron ABCD, edges AD, BD, and CD are mutually perpendicular. If AD = 2, BD = 3, and CD = 6, find the length of the altitude to the tetrahedron from point D to plane ABC.

### *Answers*

13. 10
14. 11900
15. 4/3
16. 3 or $3°$
17. 14
18. $\frac{3\sqrt{14}}{7}$ or equivalent

## Fall 1986 – Senior A – 4 – Questions

### *F86S19*

When expressed in binary notation, the number N, which is even, has nine digits. How many digits will the binary numeral for N/2 contain?

### *F86S20*

The sum of three numbers in geometric progression is 14, and the sum of their squares is 364. Find the three numbers.

### *F86S21*

The decimal numeral for the number N ends in the digit 3 (as its units digit). If the digit 3 is erased and put at the beginning of the numeral, the new numeral formed represents the number 2N. How many (decimal) digits are there in the smallest such number N?

### *F86S22*

Planet Marzipan is a perfect sphere of radius 5000 miles. Latitude $45°$ North is a circle on the surface of the planet. Compute its radius in miles. (Latitude is measured in degrees, along a circle passing through the North and South Poles of a planet. The Equator is $0°$ latitude.

### *F86S23*

If $f(x) = \dfrac{\cos 2x}{1+3\sin^2 x}$, find the smallest positive number p such that $f(x+p) = f(x)$ for all real x.

### *F86S24*

The sets $S_1$, $S_2$, $S_3$, … $S_k$ are all subsets of the set S = (1,2,3,4,5,6,7,8), and no two of the sets $S_1$, … $S_k$ have an empty intersection. What is the largest possible value of k?

### *Answers*

19. 8
20. 2, -6, 18 or 18, -6, 2
    (all three required)
21. 18
22. $2500\sqrt{2}$
23. $\pi$ or $180°$
24. 128

## Fall 1986 – Senior A – 5 – Questions

### *F86S25*

Once every five months, on the first of the month, Andy Lichtenberg unveils a new sculpture in his SoHo loft. The name he gives to the new sculpture is the name of the month in which it is unveiled. He unveils his first statute on January 1. How many months must then elapse before he has two sculptures with the same name?

### *F86S26*

If a = sin1, b = cos1, and c = tan1, where the angle is given in radian measure, which of the numbers a, b, or c is largest?

### *F86S27*

The sum of twenty different natural numbers is 210. Find the largest of these numbers.

### *F86S28*

Inside an exclusive disco are 10 men and 5 women. A line of men is outside, waiting to get in. The people from inside are chosen at random and asked to leave. If the two are of the same sex, they both leave, and one new man is admitted. If they are of opposite sexes, the woman is re-admitted and no new man is admitted. This process continues until one least person is left inside, when the disco closes. Compute the probability that this last person is a woman.

### *F86S29*

We define the new operation "*" by $a * b = a + b - ab$. If $a * b = 25$, and a and b are integers, find the largest possible value of a + b.

### *F86S30*

Larry had a network problem involving a seventh degree polynomial equation, whose roots were in arithmetic progression, some of whose roots were irrational. Going home in the rain, part of the problem became illegible. All Larry had left was that $x^7 - 28x^6 + x^4 + ... = 0$ (the coefficient of $x^5$, he knew, was zero, but he was missing the other coefficients in the equation). Nonetheless, Larry solved the equation. What was the largest root of the equation?

### *Answers*

25. 60
26. c or tan1
27. 20
28. 1
29. 25
30. $4 + 6\sqrt{6}$

## Fall 1986 – Senior A – 1 – Solutions

### *F86S1*

By direct computation, we find $496 = 2^4 \cdot 31$. The largest divisor which is a power of 2 is 16: this is too small. The next largest divisor is $2 \cdot 31 = 62$, which satisfies the conditions of the problem. The next highest divisor is 124, which is too large.

### *F86S2*

A cube has six faces, so there are $\binom{6}{2} = 15$ such segments. These segments form a regular octahedron, together with its diagonals.

### *F86S3*

We have:

$$x + y = 6$$

$$x^2 + y^2 = 10$$

Then $(x + y)^2 = 36 = x^2 + y^2 + 2xy$, so $36 = 10 + 2xy$, and $xy = 13$.

### *F86S4*

The medians of a triangle divide each other in the ratio 2:1. Thus AX = 8/3, BX = 2, and the area of triangle ABX = 8/3.

We can now relate the area of triangle ABX to that of triangle ABC. We use absolute value to denote area. Since triangles ABX, BXM have equal altitudes from B, $|ABM| = (3/2)|ABC|$. Since triangles ABM, AMC have equal altitudes from A, their areas are equal. Hence $|ABC| = 3|ABX|$, or 8.

### *F86S5*

We have 10N + 2 = 9(N + 21) = 9N + 189, so N = 187.

### *F86S6*

Note that (n + 1)! – n! = n!(n + 1 – 1) = …. Using this fact, we find that

$$\begin{aligned} N &= (10! - 9!) + (9! - 8!) + (8! - 7!) + \ldots + (2! - 1!) + 1 \\ &= 10! \end{aligned}$$

Then $MN = 10!/6! = 10 \cdot 9 \cdot 8 \cdot 7 = 5040$.

## Fall 1986 – Senior A – 2 – Solutions

### *F86S7*

Richard slept through 2/3 of the final ½ of the trip, or 1/3 of the total trip.

### *F86S8*

We will show that $\frac{1}{AC^2}+\frac{1}{CB^2}=\frac{1}{CD^2}=\frac{1}{36}$.

This result can be proved in many ways. One way is to use Ptolemy's theorem. Reflecting point C in line AB, we obtain cyclic quadrilateral ABCC'. The $AB=\sqrt{x^2+y^2}$, and Ptolemy's theorem say that $xy+xy=2CD\cdot AB$, or $xy=\sqrt{x^2+y^2}\cdot CD$. Then $\frac{1}{CD^2}=\frac{\left(x^2+y^2\right)}{x^2y^2}=\frac{1}{x^2}+\frac{1}{y^2}$.

### *F86S9*

The second car gains on the first at a rate of 10 miles per hour. The first car has a 15 mile lead, so it will take 3/2 hours for the second car to overtake the first. In this time, the second car will have traveled 82.5 miles.

### *F86S10*

Let $t=\sqrt[3]{20+14\sqrt{2}}+\sqrt[3]{20-14\sqrt{2}}$. Then

$$t^3=40+3\sqrt[3]{400-392}\left[\sqrt[3]{20+14\sqrt{2}}+\sqrt[3]{20-14\sqrt{2}}\right]$$

$$=40+3\cdot 2\cdot t$$

Hence $t^3-6t-40=0$. By inspection, 4 is a root of the equation. Division and the factor theorem show that the equation can be written as $(t-4)\left(t^2-4t+10\right)=0$. Hence the other two roots are those of the equation $t^2-4t+10=0$. The discriminant of this equation is $16-40\le 0$, so these two roots are complex, and the required value is 4.

### *F86S11*

If the number is 10t + u, then t must divide 10t + u, so it must divide u, and u must divide 10t.

| | | |
|---|---|---|
| If | t = 1 | we have: 12, 15 |
| | 2 | 24 |
| | 3 | 36 |
| | 4 | 48 |

And since t divides u, t cannot be greater than 4. There are five such numbers altogether.

### *F86S12*

It is not hard to see that MN is parallel to CD, and that triangles PMN, PAB are similar. Then MN:AB = PM:PA. In triangle PDA, angle bisector DM divides PA in the ratio

PD:DA = 10:5 = 2, so PM:PA = PN:DA = 2, and PM:PA = 2:3. Then MN = AB (2/3) = 10/3.
To see that MN is parallel to AB, we can note that triangles PAD, PBC are congruent. Then PM:MA = PD:DA (by the angle bisector theorem), or PC:CB = PN:NB, and MN must be parallel to AB.

## Fall 1986 – Senior A – 3 – Solutions

### *F86S13*

If there is only one white car, there are 9 red cars, and hence 10 blue cars. If there were two or more white cars, there would be at least 18 red cars, and there could not be any blue cars at all.

### *F86S14*

The average of three numbers in arithmetic progression is the middle number. Hence the second term of the progressions is 22. If the common difference is d, then 22(22 + d) = 528, and d = 2. Then the first term is 20, the 100th term is $22 + 99 \cdot 2 = 218$, and the sum of the first 100 terms is 100 times the average of the first term and the 100th term, or 100(20 + 218)/2 = 11900.

### *F86S15*

We have

$$(a+b)^2-(a-b)^2 = a^2+2ab+b^2-\left(a^2-2ab+b^2\right)$$
$$= 4ab = 4\left(\frac{133}{221}\right)\left(\frac{221}{399}\right) = 4\left(\frac{133}{399}\right) = \frac{4}{3}$$

### *F86S16*

Using straightedge and compass, it is easy to add, subtract, and bisect angles. Hence we can construct an angle of $15^\circ$ $(=30^\circ/2)$ and $18^\circ$, and subtract these to get an angle of $3^\circ$. If an angle of $2^\circ$ were constructible, then 18 + 2 = 20 degrees would be, so an angle of $2^\circ$ cannot be constructed. If an angle of $1^\circ$ were constructible, the an angle of 1 + 1 = $2^\circ$ would be constructible, so an angle of $1^\circ$ cannot be constructed.

### *F86S17*

Starting at t = 0, when the first two trains arrive, we can list the arrival times of the trains:
Line A:0, 16, 32, 48, 64, 80, 96, 112, 128, 144, 160, 176
Line B:0, 21, 42, 63, 84, 105, 126, 147, 168
There are 12 trains on line A and 9 on line B, but we must subtract those trains which arrive 6 minutes or less apart. There are seven of these, and 12 + 9 – 7 = 14 crowds distinguishable to the observer.

### *F86S18*

We will show that $\frac{1}{AD^2}+\frac{1}{BD^2}+\frac{1}{CD^2}=\frac{1}{DH^2}$, there H is the foot of the perpendicular from D to plane ABC (Compare problem F86S8).
Then $\frac{1}{DH^2}=\frac{1}{4}+\frac{1}{9}+\frac{1}{36}=\frac{14}{36}=\frac{7}{18}$, and $DH=\sqrt{\frac{18}{7}}=\frac{3\sqrt{14}}{7}$. We extend CH to intersect AB in R.

To get the required result, we can use the concept of the angle between two skew lines. Since two skew lines are not coplanar, they do not form an angle in the usual sense. But if we draw a third line and with one of the two skew lines, in the plane of the other skew line, we form an angle which represents the angle between the two "directions" of the skew lines.

Using this idea, we can see that $DH \perp AB$, since DH is perpendicular to the plane of points A, B, and C. Also, DC is perpendicular to AB (since DC is perpendicular to the plane of A, D, and B. Hence DH and DC are both perpendicular to AB, so plane DCHR is perpendicular to AB. This means that AB is perpendicular to line RD, since a line perpendicular to a plane is perpendicular to any line in that plane which it intersects.

From the results of problem F86S8, then $\frac{1}{DR^2} = \frac{1}{AD^2} + \frac{1}{BD^2}$.

Now since CD is perpendicular to plane ABD, triangle CDR is right-angled at D, and DH is its altitude. Hence $\frac{1}{DH^2} = \frac{1}{CD^2} + \frac{1}{DR^2}$, which gives the required result.

Note that H is the orthocenter (intersection of the altitudes) in triangle ABC. For many more solid analogues of plane theorems about right triangles, see Altshiller-Court, Modern Pure Solid Geometry, New York: Chelsea Publications, 1935, pp. 91-94. See also the Two Year College Mathematics Journal, Vol. 17, No. 1, January, 1986, p.93.

## Fall 1986 – Senior A – 4 – Solutions

### *F86S19*

An even number, in binary notation, ends in digit 0. To divide such a number by 2, we simply remove this 0.

### *F86S20*

If the middle number is a, and the common ratio r, then the numbers can be represented by a/r, a, and ar. Then:

$(i) a/r + a + ar = 14$

$(ii) a^2/r^2 + a^2 + a^2r^2 = 364$

Squaring (i), we find $a^2/r^2 + 3a^2 + a^2r^2 + 2a^2/r + 2a^2r = 196$ and subtracting (ii), we find $a^2/r + a^2 + a^2r = -84$. Since $a(a/r + a + ar) = a^2/r + a^2 + a^2r$, we can divide to get $a = -84/14 = -6$. Then $6/r + 6 + 6r = 14$, or $6(r + 1/r) = 8$, and $r + 1/r = 4/3$. It quickly follows that r = 3 or 1/3. Both values give the same progression, but in opposite order: 2, -6, 18 or 18, –6, 2.

### *F86S21*

We can form the number N by considering the multiplication algorithm. Each time we find a new digit, we can place it on the left and continue the algorithm. In this way, we find that:

$$\begin{array}{c} 157894736842105263 \\ \times 2 \\ \hline 315789473684210526 \end{array}$$

This is the smallest such number: it has 18 digits.

### *F86S22*

Taking a cross section through the poles, NS is the axis of the planet, and A and B are on its equator. If PQ is a diameter of the latitude in question, then $m\angle QOB = 45 = m\angle NOQ$, and OQ = 5000, so $XO = XQ = 2500\sqrt{2}$.

### *F86S23*

We can write f(x) as $\dfrac{1-2\sin^2 x}{1+3\sin^2 x}$. This function has the same period as the function $g(x) = \sin^2 x$, and this period is $\pi$.

### *F86S24*

Each set $A \subset S$ can be paired with its complement $A' = \{a \in S \mid a \notin A\}$. Hence we cannot choose more than half the subsets of S, since we will be forced to choose two sets which are complements (and thus have empty intersection). Since there are $2^8 = 256$ possible subsets, k cannot be more than 128.

We can get k = 128, for instance, by taking all subsets which contain the number 1 as an element. There are $2^7$ choices for the other elements of such a subset, so there are 128 such subsets altogether, all having the element in their intersection.

## Fall 1986 – Senior A – 5 – Solutions

### *F86S25*

Since 5 is relatively prime to 12, no two statutes will have the same name until all twelve names have been used up. This will happen after 12 statutes, which will take $12 \cdot 5 = 60$ months.

### *F86S26*

If $x = \pi / 4 < 1$, then sinx = cosx. Between $\pi / 4$ and $\pi / 2$, sinx increases while cosx decreases. Hence sin1 > cos1. Then tan1 = sin1/cos1 > 1, so tan1 > sin1.

### *F86S27*

Since $1 + 2 + 3 + ... + 19 + 20 = 20 \cdot 21 / 2 = 210$, the given sum is the smallest that twenty different natural numbers can have. Hence the largest of them is 20.

### *F86S28*

Each time a couple is chosen, the number of people inside the disco is reduced by one. No matter how this happens, the number of women inside remains odd. Hence the last person must be a woman (since 1 is odd), and the required probability is 1.

### *F86S29*

Since $(a-1)(b-1) = ab - a - b + 1$, we can write $a \cdot b - 1 = -(a-1)(b-1)$, which relates the new operation to ordinary multiplication. Here, we have $-(a-1)(b-1) = 24$, and we can construct a table of "* factors" of 25, based on the usual factors of 24:

| Factors of 24 | | a | b |
|---|---|---|---|
| 1 | 24 | 2 | -23 |
| | | 0 | 25 |
| 2 | 12 | 3 | -11 |
| | | -1 | 13 |
| 3 | 8 | 4 | -7 |
| | | -2 | 9 |
| 4 | 6 | 5 | -5 |
| | | -3 | 7 |

Each pair of factors of 24 will generate two pairs of "* factors" of 25, and by the symmetry of the "*" operation, we need not consider the other factorizations of 24. The table shows the maximal value of a + b to be 25.

### *F86S30*

Larry knows that the sum of the roots of the equation is 28, and since there are seven distinct roots, the average root must be 4. If d is the common difference of the arithmetic progression they form, the roots can then be written as:

$$4-3d,\ 4-2d,\ 4-d,\ 4,\ 4+d,\ 4+2d,\ 4+3d.$$

The only other clue Larry has is that the sum of the roots "taken two at a time" is zero. He can use this clue to find the sum of the squares of the roots. Since the expressions for

the roots are written symmetrically around the middle root, this will allow Larry to solve for d.
We can write

$$\left(r_1 + r_2 + r_3 + r_4 + r_5 + r_6 + r_7\right)^2 =$$
$$r_1^2 + r_2^2 + r_3^2 + r_4^2 + r_5^2 + r_6^2 + r_7^2 + 2\left(r_1 r_2 + r_1 r_3 + r_1 r_4 + \ldots + r_6 r_7\right)$$

or $\left(\sum_{i=1}^{7} r_1\right)^2 = \sum_{i}^{7} r_i^2 + 2 \sum_{1 \le i \le j \le 7} r_i r_j$ , which expresses the sum of the squares of the roots in terms of the coefficients of the equation. Substituting, we find

$$\left(\sum_{i=1}^{7} r_i\right)^2 = 2\left(16 + 9d^2\right) + 2\left(16 + 9d^2\right) + 2\left(16 + 9d^2\right) + 16$$
$$= 112 + 28d^2$$

so $28^2 = 112 + 28d^2$, and $d^2 = 168/7 = 24$, so $d = \sqrt{24} = 2\sqrt{6}$. This makes the largest root $4 + 3d = 4 + 6\sqrt{6}$ .

January 26, 1987

Dear Math Team Coach,

Enclosed is your copy of the FALL, 1986 NYCIML contests that you requested on the application form.

The following are acceptable alternative answers for the enclosed contests:

| | Question | Correct answer |
|---|---|---|
| Senior A | F86S23 | $\pi$ or $180^{\circ}$ |
| Senior B | F86B13 | 49 or 50 |

Have a great spring term.

Sincerely yours,

Richard Geller

Secretary, NYCIML

~~ONLY~~

~~complete one~~