Fall 1986 – Junior – 1 – Questions

F86J1

One half is one-third of what number?

F86J2

"How many children do you have in your family?", Mark asked Carol. Carol replied: "I forgot exactly. Bur I do remember that I have ¾ of the number of children in my family, plus ¾ of a child." How many children are in Carol's family?

F86J3

If the perimeter of a square is increased by 10x, by what percent is its area increased?

F86J4

Regular hexagon ABCDEF is inscribed in a circle of radius 12. Compute the length of diagonal AC.

F86J5

Gil traveled from point A to point B. Instead of travelling by bus, Gil got a ride in a car for half of the trip. The car went ten times as fast as Gil could have walked. For the second half of the trip, Gil had to take the bus, and traveled only twice as fast as he could have walked. How many times as fast did he complete the trip, than if he had walked the whole way?

F86J6

Tom's address is a three digit number, none of which are zero. If Tom rearranges the digits of his address, the new numbers he gets are all smaller. Bob's address is a three digit number. It contains the same digits as Tom's address, but in a different order. If Bob rearranges his digits, the only larger number he can get is Tom's address. The sum of Tom's address and Bob's address is 1233. What is Tom's address?

Answers

1. 3/2
2. 3
3. 21 or 21x
4. $12\sqrt{3}$
5. 10/3
6. 621

Fall 1986 – Junior – 2 – Questions

F86J7

The average of three numbers is x. One of three numbers is 8. Express in terms of x the sum of the other two numbers.

F86J8

If x = 1.96, compute $\dfrac{\sqrt{(x+2)^2-8x}}{\sqrt{x}-2/\sqrt{x}}$.

F86J9

In the equation “5 + 5 = 10”, the number 5 is added to itself twice to get as a sum the number 10. How many times must the number 5 be <u>added</u> to itself, to get as a sum the number 5^{25}?

F86J10

The set S consists of four prime natural numbers, and the sum of any set of three different numbers chosen from S is also a prime number. Find the smallest possible sum that <u>all four</u> elements of S could have.

F86J11

Joe put four checkers on the checkerboard illustrated at right. No row had more than one checker, and no column had more than one checker. In how many ways could Joe have arranged the checkers (reflections or rotations of the same pattern of checkers are considered distinct arrangements)?

F86J12

Circles O and P are tangent externally at point T. Chord TA and TB of circle O are extended through T to meet circle P again at points C and D respectively. If TA = 3, TB = 5, TC = 7, compute the length of TD.

Answers

7. 3x – 8
8. –1.4
9. 5^{24}
10. 48
11. 24 or 4! or equivalent
12. 35/3

Fall 1986 – Junior – 3 – Questions

F86J13

A class is raising money by selling raffles. Each raffle costs 25 cents, and the prize will be 25 dollars. The prize money is taken from the money paid for the raffles. If the class wishes to make a total profit of 25 dollars, how many raffles must the students sell?

F86J14

Find all natural numbers N which are eleven times the sum of their digits (in decimal notation). Remember that 0 is not a natural number.

F86J15

Alice picked 1/5 of the apples in an orchard. Bob then picked 1/6 of the apples left by Alice. There were 150 apples left. How many apples were in the orchard to begin with?

F86J16

Tom picked a set of natural numbers between 2 and 1000. None of Tom's numbers were prime. Each pair of Tom's numbers was relatively prime (had no common factor greater than 1). At most, how many numbers could Tom have chosen?

F86J17

If n – 1 is a multiple of 4, and n + 1 is a multiple of 5, find the remainder when n is divided by 10.

F86J18

In triangle ABC, $m\angle C = 90$, median CM = 10, and altitude CHG = 7. Find the length of the angle bisector CT.

Answers

13.	200
14.	198
15.	225
16.	11
17.	9
1.	42/5

Fall 1986 – Junior – 1 – Solutions

F86J1

If (1/3)N = 1/2, then N = 2/3.

F86J2

If Carol had N children, then 3N/4 + 3/4 = N, so 3N + 3 = 4N, and N = 3.

F86J3

If a side of the square is s, then the perimeter of the enlarged square will be (1.1)(4a) = 4.4a, so each side is 1.1a (again an increase of 10x). The area will then be $1.1(4a) = 4.4a$, which is an increase of 21x over the original area.

F86J4

Since OA = DC, the perpendicular bisector of AC is OB, so $OB \perp AC$. Also, $\overset{\frown}{BC} = 60° = \angle BOC$, and triangle OBC is equilateral. Hence triangle CMO is a 30-60-90 triangle with CO = 12, so $CM = 6\sqrt{3}$ and $AC = 12\sqrt{3}$.

F86J5

In the time Gil rode the first half of the trip, he would have walked only 1/20 of the way. In the time he rode the second half, he would have walked ¼ of the way. Hence he would have walked 6/20 of the way, and he traveled 20/6 = 10/3 times as fast.

F86J6

Suppose Tom’s address is 100a + 10b + c, where a, b, and c are decimal digits. Then a > b > c, since Tom’s address cannot be rearranged to get a larger number. Then Bob’s address must be 100a + 10c + b, and adding, we find 200a + 11(b + c) = 1233. Since b + c < 20, 11(b + c) < 220. Subtracting this from the last equation, we find that 200a > 1013, so a > 5. Also, 200a < 1233, so a < 7. Thus a = 6, 11(b + c) = 33, so b + c = 3. The only solution that fits the requirements of the problem is b = 2, c = 1.

Fall 1986 – Junior – 2 – Solutions

F86J7

If the average of the three numbers is x, then their sum is 3x. Since one of these numbers is 8, the sum of the other two must be 3x – 8.

F86J8

The numerator of the given expression is equal to $\sqrt{x^2-4x+4}$, or $\sqrt{(x-2)^2}$, which is $|x-2|$. Multiplying numerator and denominator by $\sqrt{x}$, we see that the given fraction can be written as $\frac{|x-2|\sqrt{2}}{x-2}$ or $a\sqrt{x}$, where a is 1 if x > 2 and –1 if x < 2. Hence the required value is $\sqrt{1.96}=-1.4$.

F86J9

$5^{25}=5^{24}\cdot 5=\underbrace{5+5+5+\ldots+5}_{5^{24}\,times}$.

F86J10

Trial and error is facilitated if we note that neither 2 nor 3 can be in the set S. If 2 were an element of S, the sum of 2 and two odd primes would be even. If 3 were in S, either the other three primes have the same remainder upon division by 3, or two of them have remainders of 1 and 2 respectively. In either case, the sum of three of the elements will be a multiple of 3.

Hence the smallest element of S must be at least 5. Trial and error shows that (5,7,17,19) has the desired property, and the sum of these four numbers is 48. Any other set has a larger total sum.

F86J11

If we number the rows and columns, we must match each row with exactly one column in which to place a checker. For the first row, there are four choices of columns. For the second row, there are three such choices, for the third row two choices, and one choice for the last row. Hence there are 4! = 24 possible arrangements.

F86J12

If XY is the common tangent, then $\angle ATX=\angle YTD$, since $\overset{\frown}{AT}=2\angle ATX$, $\overset{\frown}{TD}=2\angle YTD$, $\overset{\frown}{AT}=\overset{\frown}{TD}$ and $\angle B=(1/2)\overset{\frown}{AT}=(1/2)\overset{\frown}{TD}=\angle C$. Also, $\angle ATB=\angle CTD$, so triangles ATB, DTC are similar. Thus AT:TB = DT:TC, so 3:5 = 7:x, and x = 35/3.

Fall 1986 – Junior – 3 – Solutions

F86J13

Sine (.25)(100) = 25, the first 100 raffles sold will go for the prize money, and the second 100 will provide the required profit.

F86J14

Let s be the sum of the digits of N. It is not hard to see that S is usually much less than N. In fact, if N has n digits, $N \geq 10^{n-1}$, and $S \leq 9n$. Hence $11 \cdot 9n \geq 11 \cdot S = N \geq 10^{n-1}$, or $10^{n-1} \leq 99n$. A quick check shows that this is true only for n = 1,2,3. Certainly N must have more than one digit, so it has either 2 or 3 digits.

Then let N = 100a + 10b + c, where a, b, c can be ay digit (including 0). Then 100a + 10b + c = 11a + 11b + 11c, or 89a = b + 10c. Then, since a, b, c < 10, we have $89a < 9 + 10 \cdot 9 = 99$, and a = 0 or 1. If a = 0, b = c = 0, which is not a natural number. If a = 1, 89 = 10c + b, so c = 8 and b = 9.

F86J15

The 150 apples left represent 5/6 of the number Bob found there, so Bob found (6/5)(150) = 180 apples. This represents4/5 of the number Alice found originally, so there were at first (5/4)(180) = 225 apples.

This trick of working backwards arithmetically is often quicker than an algebraic solution.

F86J16

Since 1000 is between 31^2 and 32^2, and there are 11 primes which are not greater than 31, Tom could only have picked 11 numbers. Indeed, any number less than 1000 is either prime or has a prime factor less than 31 (if it had two prime factors each greater than 31, the number would be over 1000), and no two of Tom's numbers can share a prime factor. It is not hard to supply Tom with eleven numbers: just take squares of the eleven prime numbers 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31.

F86J17

Since n – 1 is even, n + is also even. An even multiple of 5 (in decimal notation) ends in a 0. Hence n + 1 ends in a 0, and n ends in a 9.

F86J18

Note first that CT bisects $\angle HCM$ as well as $\angle ACB$. This can be shown by looking at right triangle ACH and isosceles triangle CMB (CM = MB since the median to the hypotenuse of a right triangle equals half the hypotenuse). Then $\angle ACH = 90 - \angle A = \angle B = \angle MCB$, and since $\angle ACT = \angle BCT$, by subtraction $\angle HCT = \angle MCT$.

Then, in right triangle CHM, $HM^2 = CM^2 - CH^2 = 18^2 - 7^2 = 175$, so by the angle bisector theorem, HT:TM = CH:CT = 7:18, and $HT = 7/25 \cdot \sqrt{275}$. Finally, in right triangle CHT, $CT^2 = CH^2 + HT^2 = 7^2 + 49/625 \cdot 275 = 49 + 49 \cdot 11/25 = 49 \cdot 36/25$, so CT = 42/5.