New York City Interscholastic Mathematics League

SENIOR B DIVISION CONTEST NUMBER ONE

Part I Time: 11 Minutes NYCIML Spring, 1986

S86B1. Compute $\dfrac{1}{1+\dfrac{1}{1+\dfrac{1}{1+\dfrac{1}{1+1}}}}$.

S86B2. A club found that it could achieve a membership ratio of 2 men for each woman either by recruiting 24 men or by expelling x women. Compute x.

Part II Time: 11 Minutes NYCIML Sr B Contest One Spring 1986

S86B3. Compute the radius of a circle in which a 40 degree arc measures 3π units in length.

S86 B4. If $\sin\theta = 1/3$, compute $\sin^4\theta-\cos^4\theta$.

Part III Time: 11 Minutes NYCIML Sr B Contest One Spring 1986

S86B5. The infinite repeating decimal 2.7272... is written as a/b, where a and b are integers with no common factor other than 1. Compute a+b.

S86B6. Compute the degree-measure of the largest angle of a triangle whose sides have lengths 3, 5, and 7.

ANSWERS

1. 5/8	3. 27/2	5. 41
2. 12	4. -7/9	6. 120 or 120°

New York City Interscholastic Mathematics League

SENIOR B DIVISION CONTEST NUMBER TWO

Part I Time: 11 Minutes NYCIML Spring, 1986

S86B7. Diagonal AC of square ABCD is a side of square ACEF. If the area of ACEF is 8, compute the area of ABCD.

S86B8. For some real number θ, the legs of a right triangle are $\sin \theta$ and $\cos \theta$. Compute the radius of the circle circumscribing this triangle.

Part II Time: 11 Minutes NYCIML Sr B Contest Two Spring 1986

S86B9. A train leaves New York for Washington, D.C. at noon, and travels at a constant rate of 70 miles per hour. Also at noon, a train leaves Washington for New York, travelling at a constant rate of 40 miles per hour. They will meet somewhere along the route. How far away will they be one hour before they meet?

S868B10. If $\sin x + \cos x = 1/2$, compute $\sin 2x$.

Part III Time: 11 Minutes NYCIML Sr B Contest Two Spring 1986

S86B11. The integers from 1 through 10 are written on index cards. It is desired to color each of these cards with one of three colors (red, white, or blue) in such a way that any two cards with integers differing by less than three are different colors. In how many ways can this be done?

S86B12. A paper rectangle 10 units wide and 24 units long is folded and creased so that opposite vertices coincide. Compute the length of the crease formed.

ANSWERS

7. 4	9. 110 or 110 miles	11. 6
8. 1/2	10. -3/4	12. 65/6

New York City Interscholastic Mathematics League

SENIOR B DIVISION CONTEST NUMBER THREE

Part I Time: 11 Minutes NYCIML Spring, 1986

S86B13 Find a four-digit number (in decimal notation), whose two rightmost digits are identical, whose two leftmost digits are identical, and which is a perfect square.

S86B14. At a graduation party, each senior gave a snapshot of himself or herself to every other senior. If 870 snapshots were exchanged, how many seniors were at the party?

Part II Time: 11 Minutes NYCIML Sr B Contest Three Spring 1986

S86B15. Two fair six-sided dice are thrown. Compute the probability that the product of the numbers showing will be 12.

S86B16. Compute cos[Arcsin 4/5 + Arctan 5/12], where Arc denotes principal value.

Part III Time: 11 Minutes NYCIML Sr B Contest Three Spring 1986

S86B17. The measure of the interior angles of a pentagon are in arithmetic progression, with a common difference of 10 (degrees). Compute the degree-measure of the smallest angle of the pentagon.

S86B18. Find all real numbers x for which $(x+2)^4 + x^4 = 82$.

ANSWERS

13. 7744	15. 1/9	17. 88 or 88°
14. 30	16. 16/65	18. 1, -3 : both required

New York City
Interscholastic Mathematics League

SENIOR B DIVISION CONTEST NUMBER FOUR

Part I Time: 11 Minutes NYCIML Spring, 1986

S86B19. Compute $\log_{10} 4 \div \log_{10} \frac{1}{4}$.

S86B20. If a and b are positive integers such that $a^2+24=b^2$, compute the largest possible value of a+b.

Part II Time: 11 Minutes NYCIML Sr B Contest Four Spring 1986

S86B21. In parallelogram ABCD, AB = 10. Point P is the trisection point of diagonal BD which is closer to D, and AP intersects CD at Q. Compute the length of DQ.

S86B22. Two fair, six-sided dice are thrown. Compute the probability that the product of the numbers shown is <u>not</u> a prime number (remember that 1 is not a prime number).

Part III Time: 11 Minutes NYCIML Sr B Contest Four Spring 1986

S86B23. If a and b are the roots of the equation $x^2+5x+7 = 0$, compute the value of a^2+b^2.

S86B24. If $\sin^6 x + \cos^6 x = 2/3$, and $0 \leqslant x \leqslant \pi/2$, compute sin 2x.

ANSWERS

19. -1	21. 5	23.11
20. 12	22.5/6	24.2/3

New York City Interscholastic Mathematics League

SENIOR B DIVISION CONTEST NUMBER FIVE

Part I Time: 11 Minutes NYCIML Spring, 1986

S86B25. In a class of 32 students, 24 spent 3 hours each studying math, and the rest spent 2 hours each studying math. What was the <u>average</u> number of hours spent studying math by a student in the class?

S86B26. The centers of two externally tangent circles are 7 inches apart. If the length of a common external tangent to both circles is 6, compute the (positive) difference between their radii.

Part II Time: 11 Minutes NYCIML Sr B Contest Five Spring 1986

S86B27. Compute $\log_{\sqrt{2}}\left(\sqrt{2}^{\sqrt{2}}\right)^{\sqrt{2}}$.

S86B28. In rhombus PQRS, points A and B are in sides QR and RS respectively such that triangle PAB is equilateral. If PQ=PA, compute the degree-measure of angle PQR.

Part III Time: 11 Minutes NYCIML Sr B Contest Five Spring 1986

S86B29. Compute the distance between two opposite vertices of a cube whose side is 4 units long.

S86B30. Find all real values of x such that:

$$\sqrt{3x^2+6x+7}+\sqrt{5x^2+10x+14}=4-2x-x^2.$$

ANSWERS

25. 2.75 or 11/4 or equivalent	27. 2	29. $4\sqrt{3}$
26. $\sqrt{13}$	28. 80 or 80°	30. -1

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE
Senior B Division Contest Number One
Spring, 1986

S86B1. We have:

$$\frac{1}{1+\frac{1}{1+\frac{1}{1+\frac{1}{1+1}}}} = \frac{1}{1+\frac{1}{1+\frac{1}{1+\frac{1}{2}}}} = \frac{1}{1+\frac{1}{1+\frac{3}{2}}} = \frac{1}{1+\frac{1}{5/3}} = \frac{1}{1+\frac{3}{5}} = \frac{1}{8/5} = \frac{5}{8}$$

S86B2. If there are M men and W women, then M+24 =2W, or M=2W-24=2(W-12). Hence M:(W-12) = 2, and x = 12.

S86B3. A 40° arc is 40/360 = 1/9 of the circle, so $(1/9)2\pi r = 3\pi$ and 2r/9 = 3, so r = 27/2.

S86B4. $\sin^4\theta - \cos^4\theta = (\sin^2\theta+\cos^2\theta)(\sin^2\theta-\cos^2\theta) = 1\cdot(2\sin^2\theta-1) = 2(1/9) - 1 = -7/9$.

S86B5. Let N = 2.7272....
Then 100N = 272.7272....,
and 99N = 270, so N = 270/99 = 30/11, and 30+11 = 41.

S8B6. The largest angle is opposite the largest side. If this is AB, then $7^2 = 3^2+5^2-2\times3\times5\cos C$, or 49 = 34 - 30 cos C. Solving for cos C, we find cos C = -1/2, and the degree-measure of C is 120.

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE
Senior B Division Contest Number Two
Spring, 1986

S86B7. Since $AC:AB = \sqrt{2}$, and the ratio of the areas of similar figures is the square of the ratio of the sides, the ratio of the areas of the two squares is 2:1. Hence the area of ABCD is 4.

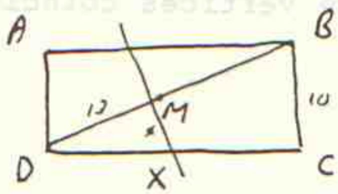
S86B8. Since $\sin^2\theta + \cos^2\theta = 1$, the hypotenuse of this triangle is 1. Since the hypotenuse is the diameter of the circumscribed circle, the radius is 1/2.

S86B9. The trains travel towards each other at the rate of 70+40 = 110 miles per hour. Hence they will be 110 miles apart one hour before they meet.

S86B10. We have $(\sin x + \cos x)^2 = \sin^2 x + \cos^2 x + 2 \sin x \cos x = 1 + \sin 2x$, so $1/4 = 1 + \sin 2x$, and $\sin 2x = -3/4$.

S86B11. There are three ways to color the first card. For each such choice, there are two ways to color the second card, making six choices in all. From these six choices, the color of each of the rest of the cards is determined.

S86B12. The crease is the perpendicular bisector of diagonal BD. To determine its length, we use similar triangles MDX, BDC. We have DB = 26, so MD = 13. If MX = x, then MD:MX = CD:BC, or 13:x = 24:10, and x = 65/12. Hence the fold is 130/12 = 65/6.



NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE
Senior B Division Contest Number Three
Spring, 1986

S86B13 Suppose the number is AABB, where juxtaposition symbolizes place value rather than multiplication. It is not hard to see that the number must be a multiple of 11, and since it contains four digits, it must be greater than 33. A quick try shows that only $88^2=7744$ works.

To see that the number must be a multiple of 11, we may write it as (AA)(100)+BB = (11)(A)(100) + (11)(B). In general, if the sum of the odd numbered digits in a decimal numeral is equal to the sum of the even numbered digits, the number represented is divisible by 11.

S86B14. If there were n seniors, then $n(n-1) = 870 = 30 \times 29$, so $n = 30$.

S86B15. The pairs of numbers which multiply to 12 are (2,6),(3,4),(4,3),(6,2). These are 4/36 = 1/9 of the possible outcomes.

S86B16. Let A = Arcsin 4/5, B = Arctan 5/12. Then sin A = 4/5, tan B = 5/12. Since both A and B are in quadrant I, cos A = 3/5, sin B = 5/13, and cos B = 12/13. Then cos (A+B) = cos A cos B - sin A sin B = (3/5)(12/13)-(4/5)(5/13) = (36-20/65) = 16/65.

S86B17. The sum of the angles is 3x180 = 540 degrees. Hence the average angle is 540/5 = 108 degrees. Since the angles are in arithmetic progression, this is the "middle" of the five angles, so the smallest is 108 -20 = 88 degrees.

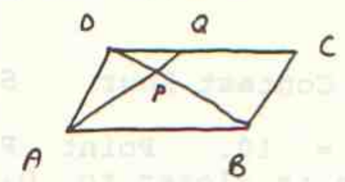
S86B18. Let $y = x+1$. Then $(y+1)^4 + (y-1)^4 = 82$. Expanding, we find many terms which cancel out, and the equation becomes $2y^4 + 12y^2 + 2 = 82$, or $y^4+6y^2+1 = 41$, or $y^4+6y^2-40 = 0$. This can be factored as a quadratic in y^2 to yield $(y^2+10)(y^2-4) = 0$, so $y^2 = -10, 4$. The root $y^2=-10$ gives no real values for x, while the root $y^2 = 4$ gives $y = 2,-2$, so $x = 1$ or -3.

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE
Senior B Division Contest Number Four
Spring, 1986

S86B19. $\log_{10} 4 \div \log_{10} \frac{1}{4} = \log_{10} 4 \div (-\log_{10} 4) = -1$.

S86B20. We have $b^2-a^2=(b+a)(b-a) = 24$. We need b+a to be the largest factor of 24 such that a and b are integers. If b+a = 24 and b-a = 1, neither a nor b are integers. If b+a = 12 and b-a = 2, then b=7 and a = 5.

S86B21. In similar triangles ABP, QDP, sides BP:DP = 2:1. Hence AB:DQ = 2:1, and DQ = 5.



S86B22. The products which produce primes are 1x2,1x3,1x5,2x1,3x1,5x1. This is 6/36 = 1/6 of the possible outcomes. This means that the product will <u>not</u> be prime 5/6 of the time.

S86B23. Write $a^2+b^2 = (a+b)^2-2ab$. Then, from the results about the sum and product of the roots of an equation, a+b=-5 and ab=7. Hence $a^2+b^2 = 25 - 2\times 7 = 25 - 14 = 11$.

S86B24. We can factor the given expression as the sum of two cubes:

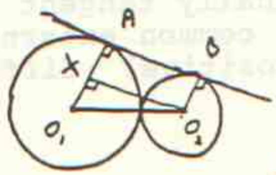
$$\sin^6 x + \cos^6 x = (\sin^2 x+\cos^2 x)(\sin^4 x - \sin^2 x\cos^2 x + \cos^4 x)$$
$$= 1[(\sin^2 x+\cos^2 x)^2 - 3\sin^2 x\cos^2 x]$$
$$= 1 - 3\sin^2 x\cos^2 x = 2/3, \text{ so}$$
$$\sin^2 x\cos^2 x = 1/9.$$

Now $\sin^2 2x = 4\sin^2 x\cos^2 x = 4/9$, and $\sin 2x = \pm 2/3$. The limits on x force us to reject the negative value.

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE
Senior B Division Contest Number Five
Spring, 1986

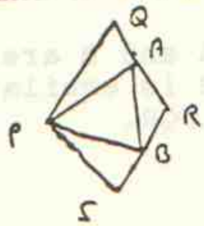
S86B25. Altogether, the students spent $24 \times 3 + 8 \times 2 = 72+16 = 88$ hours studying math. Since there are 32 students, the average time spent is $88/32 = 11/4 = 2.75$ hours.

S86B26. Drawing rectangle ABO_2X (see diagram), we find that $O_2X = AB = 6$ and $O_1O_2 = 7$. Then $O_1X^2 = 7^2-6^2 = 49-36 = 13$, so $O_1X = \sqrt{13}$. But $O_1X = O_1A - XA = O_1A - O_2B$, which is the difference in the two radii.

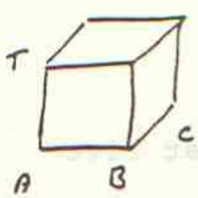


S86B27. We have: $\log_{\sqrt{2}}\left(\sqrt{2}^{\sqrt{2}}\right)^{\sqrt{2}} = \log_{\sqrt{2}}\left(\sqrt{2}^{2}\right) = \log_{\sqrt{2}} 2$. Since $2 = (\sqrt{2})^2$, $\log_{\sqrt{2}} 2 = 2$.

S86B28. If $m\angle PQR = x$, then $\angle PAQ = x$, and $\angle QPA = 180-2x$. Since angles QPS and PQR are supplementary, we find: $x+2(180-2x)+60 = 180$, or $420-3x=180$, and $x = 80$.



S86B29. In right triangle ABC, $AB = 4$ and $BC = 4$, so hypotenuse $AC = 4\sqrt{2}$. In right triangle ACT, $AC = 4\sqrt{2}$, $AT = 4$, so hypotenuse $CT = \sqrt{48} = 4\sqrt{3}$.



S86B30. We can rewrite the given equation as:

$$\sqrt{3(x+1)^2+4} + \sqrt{5(x+1)^2+9} = 5 - (x+1)^2.$$

Clearly, $\sqrt{3(x+1)^2+4} \geq \sqrt{4} = 2$, while $\sqrt{5(x+1)^2+9} \geq \sqrt{9} = 3$. Now the left side of the equation is never less than 5, while the right side is never greater than 5. Thus equality can occur only if both sides of the equation are equal to 5. This happens when $(x+1)^2 = 0$, or $x = -1$.