New York City
Interscholastic Mathematics League

Senior A Division Contest Number One Spring, 1986
Part I Time: 10 Minutes

S86S1. Compute $\dfrac{654^2-346^2}{23^2-13^2}$.

S86S2. How many multiples of three between 100 and 1000 have decimal numeral representations whose digits are all prime (1 is not a prime)?

Senior A Division Contest Number One Spring, 1986
Part II Time: 10 Minutes

S86S3. Find all real values of x for which $\log_2 |1+9/x^2| = 1$.

S86S4. An urn contains 70 marbles, which differ only in color. Twenty marbles are red, 20 are green, 20 are blue. Of the remaining ten marbles, each is either black or white, but the exact number of each color is unknown. A number of marbles are removed from the urn at random, in the dark. Even with only this incomplete knowledge of the contents of the urn, if enough marbles are removed, one can be sure that ten of them will be the same color. What is the smallest number of marbles which must be removed in order to be sure of this?

Senior A Division Contest Number One Spring, 1986
Part III Time: 10 Minutes

S86S5. The roots of the equation $x^2-px+q=0$ are $\tan\theta$ and $\cot\theta$, for some real number θ. Compute the numerical value of q.

S86S6. In triangle ABC, the degree-measures of angles A and B are 70 and 50 respectively. The altitudes of the triangle intersect at point H, and its angle bisectors intersect at point I. Compute the degree-measure of angle ICH.

ANSWERS

1. 300	3. 3, -3; both required	5. 1
2. 22	4. 38	6. 10 or 10°

New York City Interscholastic Mathematics League

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE

Senior A Division Contest Number Two Spring, 1986
Part I Time: 10 Minutes

S86S7. Squares ABCD and ABEF are adjacent faces of a cube. Find the degree-measure of angle FBD.

S86S8. The numeral 6a4 is written in base nine notation. If the number it represents is a multiple of 8, find the base nine digit a.

Senior A Division Contest Number Two Spring, 1986
Part II Time: 10 Minutes

S86S9. Compute $3^{\log_9 27}$.

S86S10. In triangle ABC, sin A:sin B:sin C = 5:7:9. Compute cos(A+B).

Senior A Division Contest Number Two Spring, 1986
Part III Time: 10 Minutes

S86S11. Compute $2^{\log_3 5} - 5^{\log_3 2}$.

S86S12. Find the smallest natural number whose cube, in decimal notation, has the three rightmost digits 432 (in that order, left to right).

ANSWERS

7. 60 or 60°	9. $\sqrt{27}$ or $3\sqrt{3}$ or equivalent	11. 0
8. 6	10. 1/10	12. 68

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE

Senior A Division Contest Number Three Spring, 1986
Part I Time: 10 Minutes

S86S13. In a certain base of numeration, $8 \cdot 8 = 54$. What is the decimal numeral for the number whose representation in this base is 84?

S86S14. If an object is released from a point above the surface of the earth and falls freely, the distance s in feet it has fallen after t seconds is given by $s = 16t^2$. Two objects are dropped at the same time, the first from a height of 100 feet and the second from a height of 76 feet. In how many seconds will the first object be twice as high from the ground as the second?

Senior A Division Contest Number Three Spring, 1986
Part II Time: 10 Minutes

S86S15. Let S be a set of natural numbers. For any natural number j, let a_j be the number of elements of S which are less than or equal to j, and let b_j be the number of elements of S which are equal to j. If $a_4 = 7$ and $a_3 = 4$, find b_4.

S86S16. The real number $\sqrt[3]{26 + 15\sqrt{3}}$ can be represented as $a + b\sqrt{3}$. Find the ordered pair of natural numbers (a,b).

Senior A Division Contest Number Three Spring, 1986
Part III Time: 10 Minutes

S86S17. Merlin the magician owns a magic purse such that any sum of money placed in the purse is doubled. Merlin charges $1.20 for each use of the purse. Simple Simon starts with a certain sum of money, puts it in the purse to double it, and, after paying Merlin his fee, reinvests all his money in the purse. After doubling his money and paying the fee, Simon reinvests all his money once more. After the money doubles and the third fee is paid, Simon finds that he has no money left at all. How much money did Simon start with?

S86S18. In equilateral triangle ABC, AB = 15. Point D is the trisection point of BC closer to B, and point E is on line AB and equidistant from points A and D. Compute the length of CE.

ANSWERS

13. 100	15. 4	17. $1.05
14. $\sqrt{13}/2$	16. (2,1)	18. 13

New York City
Interscholastic Mathematics League

Senior A Division Contest Number Four Spring, 1986
Part I Time: 10 Minutes

S86S19. In decimal notation, find the units digit of the integer

$$999^{\left(999^{\left(999^{999}\right)}\right)}.$$

S86S20. If f is a function defined on the real numbers, such that $f(x+y) + f(x-y) = 2f(x)f(y)$, and $f(1)$ is positive, compute $f(0)$.

Senior A Division Contest Number Four Spring, 1986
Part II Time: 10 Minutes

S86S21. A leg of an isosceles trapezoid is 10 units long, and a circle inscribed in the trapezoid has a radius of 2 units. Compute the number of square units in the area of the trapezoid.

S86S22. Note that $63^3 = 250047$. What is the smallest integer greater than 63 whose cube, when written in decimal notation, has a tens digit of 4 and a units digit of 7?

Senior A Division Contest Number Four Spring, 1986
Part III Time: 10 Minutes

S86S23. Five points are chosen on a circle. A zigzag path, composed of line segments, starts at any one of those points, and connects all five of them, without either intersecting itself or returning to its own starting point. How many such zigzag paths are there?

S86S24. From point P outside circle O, tangents PA and PB are drawn. A circle C is drawn tangent to AB at B and passing through point P. The two circles meet at points B and M, and AM is extended to intersect BP at X. Find the ratio BX:XP.

ANSWERS

19. 9	21. 40	23. 20
20. 1	22. 163	24. 1 or 1:1

New York City Interscholastic Mathematics League

Senior A Division Contest Number Five Spring, 1986
Part I Time: 10 Minutes

S86S25. The average of n consecutive integers, of which the smallest is n, is 94. Find n.

S86S26. If x and y are any two positive numbers, and S is the smallest of the numbers (x, y + 1/x, 1/y), compute the largest possible value of S.

Senior A Division Contest Number Five Spring, 1986
Part II Time: 10 Minutes

S86S27. If sin 2A = 1/4, find $\sin^4 A + \cos^4 A$.

S86S28 Square ABCD and ABQP are adjacent faces of a cube. Point E is the midpoint of QP. The plane containing A, C, and E intersects a fourth edge of the cube at point F. Find the area of quadrilateral ACFE if AB = 4.

Senior A Division Contest Number Five Spring, 1986
Part III Time: 10 Minutes

S86S29. Compute the value of the expression

$$\frac{\sqrt{(x+2)^2 - 8x}}{\sqrt{x} - \frac{2}{\sqrt{x}}}$$

when x = 1.21.

S86S30. Four non-coplanar points are given in space, of which no three are collinear. How many different parallelopipeds exist, of which these four points are vertices?

ANSWERS

25. 63	27. 31/32	29. -1.1
26. $\sqrt{2}$	28. 18	30. 29

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE

Senior A Division   Contest Number One   Spring, 1986

SOLUTIONS

S86S1. We have: $\dfrac{654^2-346^2}{23^2-13^2} = \dfrac{(654+346)(654-346)}{(23+13)(23-13)} = \dfrac{1000 \cdot 108}{36 \cdot 10}$

$= 100 \cdot 3 = 300.$

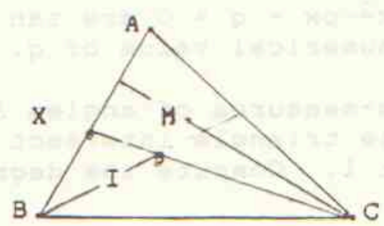
S86S2. The digits must be chosen from the set (2,3,5,7). If the number is a multiple of 3, then the sum of its digits must be a multiple of 3. Clearly, any number with three identical digits is a multiple of 3. There are four of these. If two digits are identical, they must be (2,2,5) or (2,5,5) in some order. Each of these gives three more possibilities. If the digits are all distinct, they can be 3, 5, and 7 or 2, 3, and 7. This gives twelve more possibilities. Altogether, there are 4+6+12 = 22 such numbers.

S86S3. Since x^2 is positive, $1 + 9/x^2$ must also be positive, so we may ignore the absolute value sign. Then $1 + 9/x^2 = 2$, $9/x^2 = 1$, and $x = 3$ or -3.

S86S4. We may have as many as nine red, nine blue, nine green and ten black or white marbles, without being sure of having a set of ten identically colored marbles. If one more marble is chosen, it cannot be white or black, so it must complete a set of ten marbles with the same color. Hence 38 marbles are enough.

S86S5. Since q is the product of the roots, $q = (\tan\theta)(\cot\theta) = 1$.

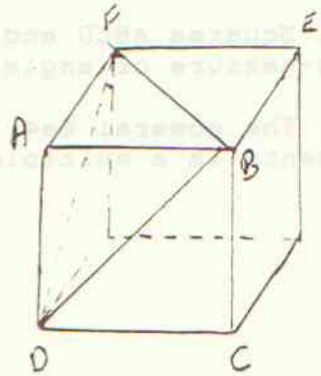
S86S6. From right triangle AXC, angle HCA = $90^\circ - \angle XAC = 20^\circ$, while ICA = $(1/2)\angle BCA = 30^\circ$. Hence $\angle ICH = \angle ICA - \angle HCA = 30^\circ - 20^\circ = 10^\circ$.



NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE

Senior A Division   Contest Number Two   Spring, 1986

SOLUTIONS



S86S7. Since triangle FBD is formed by diagonals of congruent squares, it must be equilateral. Hence $m\angle FBD = 60^{o}$.

S86S8. Using congruences, and writing in decimal notation, we have:

$$\begin{aligned} 6\cdot 81 + a\cdot 9 + 4 &\equiv 0 \bmod 8 \\ 6\cdot 1 + a\cdot 1 + 4 &\equiv 0 \bmod 8 \\ a + 2 &\equiv 0 \bmod 8 \\ a &\equiv 6 \bmod 8, \text{ so } a = 6. \end{aligned}$$

Indeed, $664_9 = 554_{10} = 8\cdot 68$ (in decimal notation).

S86S9. Let $a = \log_9 27$. Then $9^a = 27$, or $3^{2a} = 3^3$, and $a = 3/2$. The required number is $3^{3/2} = \sqrt{27}$ (or $3\sqrt{3}$).

S86S10. By the law of sines, the ratio of the sides of the triangle is 5:7:9. Since two such triangles are identical up to similarity, we can take the sides to be 5, 7, and 9. Since $A + B = 180^{o} - C$, $\cos(A+B) = -\cos C$. We use the law of cosines to compute this:

$$\begin{aligned} 9^2 &= 5^2 + 7^2 - 2\cdot 5\cdot 7\cos C \\ 81 &= 74 - 70 \cos C \\ 7 &= -70 \cos C \\ \cos C &= -1/10, \text{ and } \cos(A+B) = 1/10. \end{aligned}$$

S86S11. Let $x = 2^{\log_3 5}$. Then $\log_5 x = \log_3 5\cdot\log_5 2 = \log_3 2$, so

$5^{\log_3 2} = x$, and $2^{\log_3 5} - 5^{\log_3 2} = x - x = 0$.

S86S12. If N is the number we want, it is not hard to see that the units digit of N must be 8, so $N^3 = (10k+8)^3$ for some natural number k. Then $n^3 = 1000k^3 + 2400k^2 + 1920k + 512 = 100m + 10(2k+1) + 2$. If the tens digit of N^3 is 3, then $2k+1 = 10t+3$ or $10t+13$ for some integer t. The smallest such values of k are 1 or 6. A quick check shows that $68^3 = 314432$.

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE

Senior A Division Contest Number Three Spring, 1986

SOLUTIONS

S86S13. If the base is b, then (writing all numbers in decimal notation), $8 \cdot 8 = 64 = 5b + 4$, so $b = 12$. In base 12, $84 = 8 \cdot 12 + 4 = 100$.

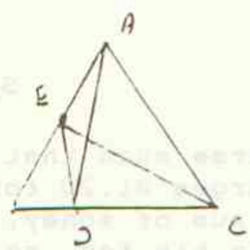
S86S14. We have
$$2(76 - 16t^2) = 100 - 16t^2,$$
$$152 - 32t^2 = 100 - 16t^2,$$
$$16t^2 = 52$$
$$t = \sqrt{52} = \sqrt{13}/2$$

S86S15. Since 7 elements are less than or equal to 4, while only 4 are less than or equal to 3, there must be four elements equal to 4, and $b_4 = 4$.

S86S16. Let $26+15\sqrt{3} = (a+b\sqrt{3})^3 = a^3+3\sqrt{3}b^3+ 3a^2b\sqrt{3} + 9ab^2$. Then:
(i) $a^3+ 9ab^2= 26$, and
(ii) $3b^3+ 3a^2b = 15$, or $b(b^2+a^2) = 5$.
By inspection of (ii), b=1 and a = 2. This ordered pair satisfies equation (i) as well.

S86S17. If Simon started with x dollars, he first doubled it to 2x. After paying Merlin, he has 2x-\$1.20 dollars. Doubling and paying gives him 4x-\$3.60. A final doubling and paying gives him 8x - \$8.40. Since this exhausts his funds, 8x - \$8.40 = 0, and x = \$1.05.

S86S18. Let AE = ED = x. Then BE = 15 - x, and using the law of cosines in triangle EBD, we find $x^2 = 25 + (15-x)^2 - 5(15-x) = 175 - 25x + x^2$, so x = 7. Letting CE = y, we can use the law of cosines in triangle AEC to find that $y^2 = 7^2+ 15^2- 7 \cdot 15 = 274 - 105 = 169$, and y = 13.



NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE

Senior A Division — Contest Number Four — Spring, 1986

SOLUTIONS

S86S19. Let the given number be N. Then $N = 999^k$, for some odd integer k. Now an odd power of 9 ends in a 9, so N must also end in a 9.

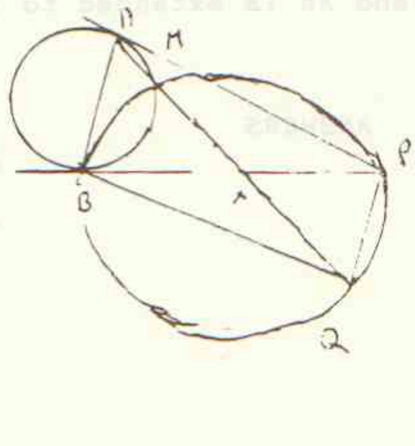
S86S20. Letting $x = y = 0$, we have $2f(0) = 2[f(0)]^2$, or $f(0) = [f(0)]^2$, so that $f(0) = 0$ or 1. If $f(0) = 0$, then, for any real, we can let $y = 0$. Then we find that $f(x) + f(x) = 2f(x) = 0$, so $f(x) = 0$ for any value of x. In particular, $f(1) = 0$, which is contrary to the conditions of the problem. Hence $f(0) = 1$.

S86S21. The area of any polygon circumscribed about a circle is rs, where r is the radius of the circle and s is half the perimeter of the polygon (the proof is similar to the proof of the same formula for a triangle, or to the proof that the area of a regular polygon is half the apothem times the perimeter). Also, if a quadrilateral has an inscribed circle, then the sums of the lengths of opposite sides are equal. Hence the semiperimeter of the given figure is 20, and the area is 40 square units.

S86S22. Suppose that N^3 is of the required form. By examining 1^3, 2^3, 3^3,... we see that the units digit of N is 3, so we can write $N^3 = (10a+3)^3 = 1000a^3 + 900a^2 + 270a + 27$. Clearly the first two terms contribute nothing to the last two digits, so we need $270a + 27 = 100x + 47$ for some integer x, or $27a = 10x + 2$. Hence, for the value of a to be correct we need a multiple of 27 with a units digit of 2. By inspection, a = 6, 16, 26, etc. and $N = 10a + b = 63, 163, 263$, etc.

S86S23. We have five choices for points to start our zigzag path. If we start it at some point P, we can choose two points to visit next in forming our line (if we choose any point other than P's two neighbors, we will cut off some points from others, so that the zigzag path will have to intersect itself). Similarly, after this choice, we still have two choices for the next vertex to visit, and two choices for a third vertex. After the fourth vertex, we have no choice for the last vertex. Hence we have $5 \cdot 2 \cdot 2 \cdot 2$ paths to choose from. But this counts each path twice (traversed both forward and backward), so there are actually only $5 \cdot 2 \cdot 2 = 20$ choices.

S86S24. Let AX intersect circle C again at Q. We will show that APQB is a parallelogram, so that BX:XP = 1. If we draw BM, we can let $\overset{\frown}{BM} = a$ in circle O and $\overset{\frown}{BM} = b$ in circle P. Then $m\angle MAB = a/2 = m\angle MBP = m\angle MQP$, so $AB \parallel PQ$. Similarly, $m\angle MQB = b/2 = m\angle ABM = AM/2 = m\angle PAQ$, so $AP \parallel BQ$.



NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE

Senior A Division Contest Number Five Spring, 1986

SOLUTIONS

S86S25. The average of a set of consecutive integers is also the average of the smallest and largest element of the set. Here, the smallest is n and the largest is n + n - 1 = 2n-1. Hence 94 = (3n-1)/2, 188 = 3n-1, and n = 63.

S86S26. Let S_{max} = a. Then x >= a, 1/y >= a, so 1/x <= 1/a, y <= 1/a, and y + 1/x <= 2/a. But y + 1/x >= a, so a <= 2/a, or a <= $\sqrt{2}$. If x = $\sqrt{2}$, y = 1/$\sqrt{2}$, then it is easy to see that S actually assumes the value $\sqrt{2}$.

S86S27. We have $1 = 1^2 = (\sin^2 A + \cos^2 A)^2$

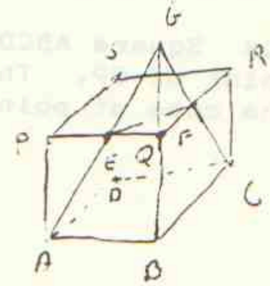
$= \sin^4 A + 2\sin^2 A\cos^2 A + \cos^4 A$,

so $\sin^4 A + \cos^4 A = 1 - 2\sin^2 A\cos^2 A = 1 - (1/2)\cdot 4\sin^2 A\cos^2 A$

$= 1 - (1/2)(\sin 2A)^2 = 1 - (1/2)\cdot(1/16) =$

$= 1 - 1/32 = 31/32.$

S86S28. Let AE intersect CF at G, and let E be the intersection of plane ACF with PQ. Let U, T be the feet of perpendiculars from G to EF, AC respectively. We use the following facts about the diagram:



(i) G is on line BQ

(ii) GQ:GB = GE:GA = GF:GC = GU:GT = 1:2

(iii) PE = EQ.

Then AC = 4r2, EF = 2r2, and the desired area is (1/2)UT(EF+AC) = (1/2)UT(6r2) = (1/4)GT(6r2). Using the Pythagorean theorem in triangle GTB, we find $GT^2 = TB^2+GB^2$ = 8 + 64 = 72, so GT = r72 = 6r2, and UT = 3r2. The desired area is (1/4)(6r2)(6r2) = (1/4)·6·2·6 = 18.

Property (i) above follows from the fact that G is on AE, hence on plane ABQ and also on line CG, hence on plane EQB. Thus point G is on the line of intersection of the two planes. Property (ii) above follows from various similar triangles in the figure: the top and bottom faces of the cube, together with a plane parallel to them through G, cut any transversal line into two equal parts. Property (iii) follows from the similarity of triangles EFG, ABC.

S86S29. We have:

$$\frac{\sqrt{(x+2)^2-8x}}{\sqrt{x}-\frac{2}{\sqrt{x}}} = \frac{\sqrt{x^2-4x+4}}{\sqrt{x}-\frac{2}{\sqrt{x}}} = \frac{|x-2|}{x-2}\cdot\sqrt{x}$$

For any non-zero real number a, |a|/a = 1 or -1, depending on whether a is positive or negative. Here, x-2 <0, and $\sqrt{x}$ = 1.1, so the value of the expression in -1.1.

586S30. Choose any two of the points, say A and B. These points can be (a) endpoints of an edge, or (b) endpoints of a diagonal of a face, or (c) endpoints of a 'body diagonal' of the parallelopiped.

Case (a): (i) Suppose a third point, say C, lies on the same face as A and B. Then (since A and B are endpoints of an edge), there are two possible parallelograms with A, B, and C as vertices. For each of these parallelograms, point D can be the endpoint of an edge from any of the four vertices. This gives 8 possible parallelopipeds. (ii) Suppose that neither of the remaining points C or D lies on a face of the solid containing AB. The C and D can only be endpoints of an edge which is parallel to AB, and A, B, C, and D would be coplanar. This is contrary to assumption.

Case (b): AB is a diagonal of a face of the parallelogram. (i) If a third point C lies on a face with AB, then there are four new parallelopipeds possible. (ii) If C and D are not on the same face as AB, then CD is either a diagonal of a face opposite AB (in which case exactly one new parallelopiped is possible) or an edge of such a face (in which case four new parallelopipeds are possible).

Case (c): AB is a 'body diagonal' of the parallelopiped. Then C and D lie on the same face as either A or B; that is, three of the four given points lie on the same face of the parallelopiped. We can choose this face in three different ways, and the fourth vertex can be the other end of an edge which begins at any one of the other three vertices. This gives 3x4 = 12 new parallelopipeds.

Thus, altogether there are 8 + 4 + 1 + 4 + 12 = 29 parallelopipeds possible.

May 12, 1986

Dear Math Team Coach,

Enclosed is your copy of the Spring, 1986 NYCIML contests that you requested on the application form.

The following are corrections for the enclosed contests:

	Question	Correct answer
Senior A	S86S1	7700/9
	S86S15	This question was poorly worded and was eliminated from the competition.
	S86S23	20 or 40

Have a great summer.

Sincerely yours,

Richard Geller

Secretary, NYCIML

NO JR