New York City Interscholastic Mathematics League

Senior A Division Contest Number One Fall, 1985
Part I Time: 10 Minutes

F85S1. The sum of four different prime numbers divides their product. Find the smallest of these prime numbers.

F85S2. What is the first time after 6:00 that the hands of a clock are perpendicular?

Senior A Division Contest Number One Fall, 1985
Part II Time: 10 Minutes

F85S3. In a circle, chord AB = 24. Point M is the midpoint of minor arc AB, and the length of chord AM is 13. Find the radius of the circle.

F85S4. Compute the numerical value of $\log_{\left(3^{\log_3 9}\right)} 81$.

Senior A Division Contest Number One Fall, 1985
Part III Time: 10 Minutes

F85S5. Find the largest real number x such that:

$$\sqrt{x - \sqrt{2x - 1}} + \sqrt{x + \sqrt{2x - 1}} = \sqrt{2}.$$

F85S6. If Arctan denotes principal value, find tan (4 Arctan 1/3).

ANSWERS

1. 2
2. 6:16 4/11 or equivalent
3. 119/10 or equivalent
4. 2
5. 1
6. 24/7

New York City Interscholastic Mathematics League

Senior A Division Contest Number Two — Fall, 1985

Part I Time: 10 Minutes

F85S7. From point P, outside circle O, lines PX and PY are drawn tangent to the circle at points X and Y respectively. If PO = PX + PY, find the degree-measure of angle XPY.

F85S8. If $f(x) = x^2 - 6x + 10$, $g(x) = x^2 + ax + b$, and the roots of the equation $g(x) = 0$ are the squares of the roots of the equation $f(x) = 0$, find the ordered pair of real number (a,b).

Senior A Division Contest Number Two — Fall, 1985

Part II Time: 10 Minutes

F85S9. Find the radius of the circle whose center is at the origin of a rectangular coordinate system, and which is tangent to the line $x + 2y = 10$.

F85S10. If a and b are decimal digits, the decimal numeral a975b is a multiple of 88. Find the ordered pair (a,b).

Senior A Division Contest Number Two — Fall, 1985

Part III Time: 10 Minutes

F85S11. Compute the numerical value of $\sin^2 \pi/10 + \sin^2 2\pi/5$.

F85S12. In regular tetrahedron ABCD, points M and N are midpoints of sides BC and AD respectively. The plane through A, M and D intersects the plane through B, N, and C in a line. If BC = 1, find the length of that part of the line of intersection which is inside tetrahedron ABCD.

ANSWERS

7. 120 or 120°

8. (-16,100); ordered pair required.

9. $2\sqrt{5}$

10. (5,2); ordered pair required

11. 1

12. $\sqrt{2}/2$

New York City
Interscholastic Mathematics League

Senior A Division Contest Number Three — Fall, 1985

Part I Time: 10 Minutes

F85S13. Compute the numerical value of $2(\log_2 2^2)^2$.

F85S14. If $a_1 = 400$, and a_n is the sum of the cubes of the (decimal) digits in a_{n-1}, (for n = 1,2,3,...), find a_{1985}

Senior A Division Contest Number Three — Fall, 1985

Part II Time: 10 Minutes

F85S15. The lengths of the sides of a certain rectangle are all integers, and the number of square units in its area is one less than the number of linear units in its perimeter. Find the area of the rectangle.

F85S16. In triangle ABC, AB = AC. Point D is a point on line segment AC for which AD = DB = BC. Find the degree-measure of angle A.

Senior A Division Contest Number Three — Fall, 1985

Part III Time: 10 Minutes

F85S17. For some natural number n, the integer $n^2 + 2n$, when written in decimal notation, has a units digit of 4. What is the tens digit of this integer (in decimal notation)?

F85S18. Two candles of the same length are lit at the same time. One candle burns down uniformly in four hours, and the other in five hours. How many hours after the candles are lit will one candle be five times as long as the other?

ANSWERS

13. 8	15. 15	17. 2
14. 250	16. 36 or 36°	18. 80/21 or equivalent.

New York City Interscholastic Mathematics League

Senior A Division Contest Number Four — Fall, 1985
Part I Time: 10 Minutes

F85S19. In decimal notation, what is the units digit of the integer
$1! + 2! + 3! + 4! + 5! + . . . + 8! + 9! + 10!$?

F85S20. In square ABCD, point M is the midpoint of side BC, and point N is on side CD such that NM is perpendicular to AM. Find the ratio CN:ND.

Senior A Division Contest Number Four — Fall, 1985
Part II Time: 10 Minutes

F85S21. If $\dfrac{3f(x)}{f(x)-3} = x$ for all real numbers x, find f(6).

F85S22. Two tiles with the letter T and two with the letter O are placed in an urn. A tile is withdrawn and placed on a rack. This action is repeated until the rack contains an arrangement of all four letters. What is the probability that the two O's will be separated and the two T's will also be separated?

Senior A Division Contest Number Four — Fall, 1985
Part III Time: 10 Minutes

F85S23. If $0 < x < \pi/2$, and $\cos^4 x + \sin^4 x = 3/4$, find the radian measure of x.

F85S24. In quadrilateral ABCD, opposite sides AB and CD are perpendicular to each other, and each is 10 units long. Circles drawn on each of these sides as diamters are tangent to each other. If BC = 7 AD, find the area of quadrilateral ABCD.

ANSWERS

19. 3
20. 1:3 or equivalent
21. 6
22. 1/3.
23. $\pi/8$
24. 72

New York City Interscholastic Mathematics League

Senior A Division Contest Number Five — **Fall, 1985**
Part I Time: 10 Minutes

F85S25. The inhabitants of a certain kingdom will always tell the truth when asked when their birthday (the anniversary of their birth) is, except if they are asked on their actual birthday. One January 5, Mr. Aardvark (one such inhabitant) was aked when his birthday was. He replied, 'It was yesterday'. On January 6, he was asked the same question, and gave the same answer. On what day in January was he born?

F85S26. Find the largest prime number which is four more than the fourth power of an integer.

Senior A Division Contest Number Five — **Fall, 1985**
Part II Time: 10 Minutes

F85S27. Circles O_1 and O_2 intersect at points P and Q. Line O_1P intersects circle O_2 at points P and X, and line O_2P intersects circle O_1 at points P and Y. If the measure of arc PQ in circle O_1 is 100°, find the degree-measure of angle XYP.

F85S28. Express in radical form $\tan 18^\circ + \tan 42^\circ + \sqrt{3}\tan 18^\circ \tan 42^\circ$.

Senior A Division Contest Number Five — **Fall, 1985**
Part III Time: 10 Minutes

F85S29. Circle O is inscribed in isosceles triangle ABC. Points M and N are midpoints of legs AB and AC respectively, and MN is tangent to circle O. Compute the ratio AB:BC.

F85S30. If $f_1(x) = \frac{x\sqrt{3} - 1}{x + \sqrt{3}}$, $f_2(x) = f_1(f_1(x))$, $f_3(x) = f_1(f_2(x))$, and $f_n(x) = f_1(f_{n-1}(x))$ for $n > 1$, find $f_{1986}(1985)$.

ANSWERS

25. January 5 or 1/5 or equivalent. Do not accept the ambiguous 'yesterday'!

26 5

27. 50 or 50°

28 $-\sqrt{3}$

29. 3:2 or equivalent

30 1985

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE

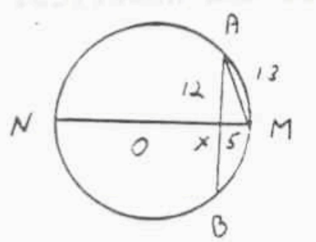
Senior A Division Fall, 1985

SOLUTIONS

F85S1. If all the primes were odd, then their sum would be even. Since the sum divides the product, the product would also have to be even. But the product of four odd numbers is odd, so not all of the primes are odd. The only even prime is 2, which must be the smallest of the four in the problem. An example of a set of primes having the property described is {2,3,11,17}.

F85S2. At 6:00, the hands form an angle of 180°. At r minutes after 6, the minute hand has moved through $(r/60)\cdot 360 = 6r$ degrees, while the hour hand has moved through $(r/60)\cdot 30 = r/2$ degrees. Hence, if r minutes have elapsed since 6:00 (and the minute hand has not passed the hour hand), the angle between the hands is $180 - 6r + r/2$ degrees. If this angle is 90°, then $180\ 11r/2 = 90$, and $11r/2 = 90$, or $r = 180/11 = 16\ 4/11$.

F85S3. Triangle AMX is a 5-12-13 triangle (see diagram), and the products of segments of intersecting chords are equal, so $5(2r-5) = 12\cdot 12 = 144$, and $10r-25=144$, so $r=199/10$.



F85S4. Since $3^{\log_3 9} = 9$, the given expression equals $\log_9 81 = 2$.

F85S5. Squaring, we find $2x + 2\sqrt{x^2 - (2x-1)} = 2$

$$2x + 2\sqrt{(x-1)^2} = 2$$

$$2x + 2|x-1| = 2$$

For $x \geq 1$, this gives $4x-2 = 2$, and $x = 1$.
For $x < 1$, $2x-2x+2 = 2$, so any number in this range (which makes the radicals meaningful) will do. Since the largest x is called for, $x = 1$.

F85S6. Let $x = \text{Arctan } 1/3$. Then we want $\tan 4x$.

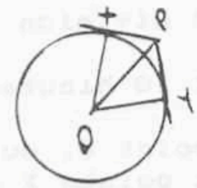
$$\tan 2x = \frac{2\tan x}{1-\tan^2 x} = \frac{2/3}{1-1/9} = \frac{2}{3}\cdot\frac{9}{8} = \frac{3}{4}$$

$$\text{and } \tan 4x = \frac{2\tan 2x}{1-\tan^2 2x} = \frac{3/2}{1-9/16} = \frac{3}{2}\cdot\frac{16}{7} = \frac{24}{7}$$

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE

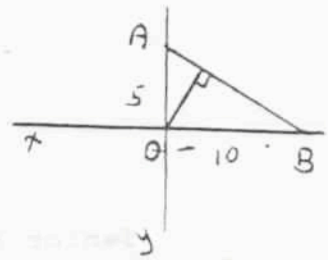
Senior A Division Contest Number Two Fall, 1985

SOLUTIONS



F85S7. In right triangle PXO, PO = 2PX. Hence $m\angle XOP = 30^\circ$, $m\angle XPO = 60^\circ$, and $m\angle XPY = 120^\circ$.

F85S8. Let $y = x^2$. We want a quadratic equation satisfied by y. Certainly $y - 6\sqrt{y} + 10 = 0$, or $y + 10 = 6\sqrt{y}$. Squaring, we find $y^2 + 20y + 100 = 36y$, or $y^2 - 16y + 100 = 0$. This is the required equation.

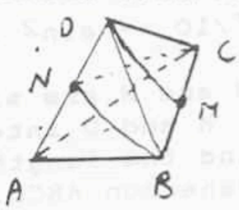


F85S9. The x- and y- intercepts of the given line (see diagram) are 10 and 5 respectively, so the length of AB is $5\sqrt{5}$. Computing the area of triangle in two ways (or otherwise), we find that the length of the altitude to the hypotenuse is $2\sqrt{5}$. This is also the radius of the circle centered at the origin which is tangent to the line.

F85S10. The number must be a multiple of 8 and of 11. To be a multiple of 8, the number formed by the three rightmost digits must be multiple of 8. Checking, we find that only b = 2 satisfies this condition. To be a multiple of 11, the 'alternate sum' of the digits must be a multiple of 11, or $2 - 5 + 7 - 9 + a = a-5 = 0$ (since zero is the only multiple of 11 which is five less than a decimal digit). This requires a = 5. Checking, 59752 = 88x679.

F85S11. Since $2\pi/5 + \pi/10 = 5\pi/10 = \pi/2$, $\sin 2\pi/5 = \cos \pi/10$. Hence the given expression is equal to $\sin^2\pi/10 + \cos^2\pi/10 = 1$.

F85S12. Since plane AMD contains line AD, it contains point N, and hence it contains line segment MN. A similar argument shows that plane BCD also contains segment MN, so it is the length of this segment that is required. To find it, we can look at right triangle AMN, in which AN = 1/2 and AM = $\sqrt{3}/2$ (since AM is an altitude in equilateral triangle ABC). The Pythagorean theorem then shows that MN = r(3/4)(1/4) = $\sqrt{2}/2$.



NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE

Senior A Division Contest Number Three Fall 1985

SOLUTIONS

F85S13. We have $2(\log_2 2^2)^2 = 2(2\log_2 2)^2 = 2(2\cdot 1)^2 = 2\cdot 4 = 8$.

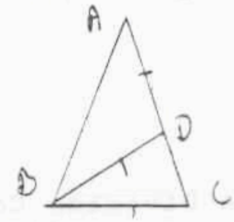
F85S14. By direct computation, we have

$$a_2 = 64, \quad a_3 = 280, \quad a_4 = 520, \quad a_5 = 133, \quad a_6 = 55, \quad a_7 = 250, \quad a_8 = 133,$$

and the cycle 133,55,250 will continue repeating. Hence $133 = a_5 = a_8 = a_{11} = \ldots = a_{3n-1}$, for all $n > 1$. Since $1985 = 3\cdot 661 - 1$, $a_{1985} = 133$.

F85S15. By trying small integers, it is not hard to see that the dimensions of the rectangle are 3x5, so that its area is 15. To show that these are the only possible dimensions, let them be a and b. Then $ab = 2a+2b-1$. If we let $a+b = q$, then $ab = 2q-1$, and a and b are roots of the equation $x^2 - qx + 2q-1 = 0$. For the roots to be rational it is necessary that the discriminant be a perfect square. This condition can be written as is $q^2-8q+4 = (q-4)^2-12 = N^2$ for some integer N, or $(q-4)^2-N^2 = 12$, or $(q-4+N)(q-4-N) = 12$. Comparing factors of the number 12 with the expression on the left, we find that q can only be 8, which gives a,b = 5,3.

F85S16. Let $m\angle A = a$. Then triangle ADB is isosceles, so $m\angle ADB = 180-2a$. Since triangle BDC is also isosceles, $m\angle C = m\angle BDC = 180 - m\angle BDA = 2a$. Then, in triangle ABC, $\angle A + \angle B + \angle C = a + 2a + 2a = 180$, so $a = 36^\circ$.



F85S17. Since $n^2 + 2n$ ends in the digit 4, $n^2 + 2n + 1 = (n+1)^2$ ends in the digit 5. Since this number is a perfect square, its tens digit must be a 2. This is the tens digit of the number in the problem as well.

F85S18. If we think of the flame travelling down each candle at a certain rate, this becomes a familiar sort of problem, which can be 'diagrammed' as below. The length of one candle is taken as the unit of length, and time is expressed in hours.

	Distance	Rate (in candle-lengths per hour)	Time
Slow flame	t/5	1/5	t
Fast flame	t/4	1/4	t

The distances compared in the problem are those that remain for the flame to travel, so $1 - t/5 = 5(1 - t/4)$, which gives $t = 80/21$ hours.

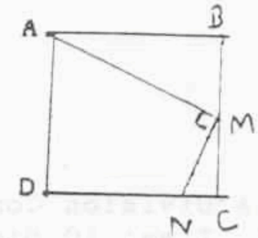
NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE

Senior A Division   Contest Number Four   Fall 1985

SOLUTIONS

F85S19. Since 5!, 6!, 7!, . . . 10! all contain factors of 2 and 5, they all have units digits equal to 0. The units digit required must come from the sum 1! + 2! + 3! + 4! = 1 + 2 + 6 + 24 = 33.

F85S20. From similar triangles ABM, NCM, we find AB:BM = MC:CN = 2:1, so NC = (1/2)MC = (1/4)BC. Since BC = CD, NC:ND = 1:3.



F85S21. Solve the relation in the problem for $f(x)$:

$$3f(x) = xf(x) - 3x$$
$$(3 - x)\,f(x) = -3x$$
$$f(x) = 3x/(x-3).$$

Thus $f(6) = 18/3 = 6$.

F85S22. For any first choice, the probability that the second choice will not match is 2/3. For the experiment to 'succeed', the remaining two letters must be chosen in the same order as the first two. This will happen 1/2 of the time. Hence the probability of success is (2/3)(1/2) = 1/ .

F85S23. We have $\cos^4 x + \sin^4 x = (\cos^2 x + \sin^2 x)^2 - 2\cos^2 x \sin^2 x$
$= 1 - (1/2)\sin^2 2x = 3/4$,

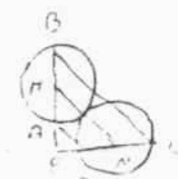
so $\sin^2 2x = 1/2$. Since x is positive and acute, $\sin 2x = \sqrt{2}/2$, $2x = \pi/4$, and $x = \pi/8$.

F85S24. Let AB intersect CD at P (see diagram), and let M, N be midpoints of AB, CD respectively. Let AP = x, PD = y. Then $AD^2 = x^2+y^2$, and 44 $AD^2 = (x+10)^2+(y+10)^2 = BC^2$. Thus we have:

$$\text{(i)}\quad (x+5)^2+(y+5)^2 = 100 \qquad [\text{since } MN^2 = PM^2+PN^2]$$
$$\text{(ii)}\quad 49(x^2+y^2) = (x+10)^2+(y+10)^2.$$

From (i), $x^2+10x+y^2+10y = 50$, while from (ii), $48x^2-20x+48y^2-20y = 200$. Eliminating x^2 and y^2, we find $500x + 500y = 2200$, or $x+y = 22/5$. We can avoid solving for x and y separately by computing the area ABCD as the difference of the areas APD, BPC:

$$\text{Area}(ABCD) = (1/2)(x+10)(y+10) - (1/2)xy = 5x+5y+50 = 72$$

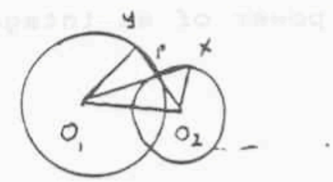


SOLUTIONS

F85S25. Aardvark could not have been born on January 4, since then his answer on January 6 (which was not his birthday) would have been 'the day before yesterday'. Hence he must have been lying on January 5, which is thus his birthday.

F85S26. Let $p = n^4+4 = n^4+4n^2+4-4n^2$
$= (n^2+2)^2-(2n)^2$
$= (n^2+2n+2)(n^2-2n+2)$.
Thus the number p cannot be prime unless one of these factors is 1. This happens if n = 1 or -1, and p must equal 5.

F85S27. Since isosceles triangles O_1PY, O_2PX have base angles YPO_1, XPO_2 equal, these triangles must be similar. Thus angles O_1YO_2 and O_1XO_2 are equal, and quadrilateral O_1YXO_2 can be inscribed in a circle. In this new circle, angles XYO_2 and XO_1O_2 intercept the same arc, so these angles are equal. Since the latter is a central angle in circle O_1, it is easily seen to measure 50°.



F85S28. We use the fact that 18 + 42 = 60.

$$\sqrt{3} = \tan 60^\circ = \tan(18^\circ + 42^\circ) = \frac{\tan 18^\circ + \tan 42^\circ}{1 - \tan 18^\circ \tan 42^\circ}, \text{ so}$$

$\tan 18^\circ + \tan 42^\circ = \sqrt{3}(1 - \tan 18^\circ \tan 42^\circ)$, and the complete expression of the problem must equal $\sqrt{3} - \sqrt{3}\tan 18^\circ \tan 42^\circ + \sqrt{3}\tan 18^\circ \tan 42^\circ = \sqrt{3}$.

F85S29. Let AB = a, BC = b. Since quadrilateral BCMN is circumscribed about a circle, MN+BC=MB+NC. Since MN = BC/2, we have 3b/2=a, and a:b = 3:2.

F85S30. Method I: By direct calculation, we have $f_2(x) = \frac{x-\sqrt{3}}{\sqrt{3}x-1}$, $f_3(x) = -1/x$. Hence $f_6(x) = f_3(f_3(x)) = x$. Since 1986 = 6*331, $f_{1986}(x) = x$, and $f_{1986}(1985) = 1985$.

Method II: More generally, we can consider functions of the form $f(x) = \frac{ax+b}{-bx+a}$ where a and b are real numbers. It is not hard to verify that the composition of these functions acts exactly like multiplication of the complex number a+bi (the systems are isomorphic). Using this idea, we can write:

$$f_1(x) = \frac{x\frac{\sqrt{3}}{2} - \frac{1}{2}}{\frac{x}{2} + \frac{\sqrt{3}}{2}}$$

so that f_1 corresponds to the complex number $\frac{\sqrt{3}}{2} - \frac{1}{2}i$, and f_{1986} corresponds to the number $(\frac{\sqrt{3}}{2} - \frac{1}{2}i)^{1986} = (\text{cis}(-30^\circ))^{1986}$. Using De Moivre's theorem gives the desired result.

January 5, 1986

Dear Math Team Coach,

Enclosed is your copy of the Fall, 1985 NYCIML contests that you requested on the application form.

The following are corrections for the enclosed contests:

Senior A	Question	
	F85S3	Answer is 169/10
	F85S14	Answer is 133
	F85S23	Answer $\pi/8$ and $3\pi/8$
	F85S24	The question is impossible and was eliminated from the contest.
Senior B	F85B21	The question is poorly worded and either 256 or 1024 were accepted.

Have a good year.

Sincerely yours,

Richard Geller

Secretary, NYCIML