# New York City Interscholastic Mathematics League

Junior Division Contest Number One — Fall, 1985
Part I Time: 10 Minutes

F85J1. How many integers between 1000 and 5000 are perfect squares?

F85J2. If $x+y = 34$ and $y-z = 12$, compute the value of $x+z$.

Junior Division Contest Number One — Fall, 1985
Part II Time: 10 Minutes

F85J3. The legs of a right triangle are 4 and 6. Find the length of the altitude to the hypotenuse of the triangle.

F85J4. A train begins crossing a bridge 450 meters long at noon, and clears the bridge completely 45 seconds later. A stationary observer notices that the train takes 15 second to pass him. Compute the rate of the train in meters per second.

Junior Division Contest Number One — Fall, 1985

Part III Time: 10 Minutes

F85J5. Find all real values of x for which $2^{(x^2)} = (2^x)^2$.

F85J6. In trapezoid ABCD, M and N are midpoints of legs AD, BC respectively, and line MN intersects diagonal AC at P and diagonal BD at Q. If AB = 7 and CD = 13, compute the length of PQ.

ANSWERS

1. 39
2. 22
3. $12/\sqrt{13}$ or equivalent
4. 15 or 15 meters per second.
5. 0, 2: both required
6. 3

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NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE

Junior Division Contest Number Two — Fall, 1985
Part I Time: 10 Minutes

F85J7. Find the smallest positive integral multiple of 12 which leaves a remainder of 4 when divided by 52.

F85J8. In triangle ABC, BC = 10. Points X and Y are on line segments AB, AC respectively, and XY||BC. If the area of triangle AXY is half that of triangle ABC, compute the length of XY.

Junior Division Contest Number Two — Fall, 1985
Part II Time: 10 Minutes

F85J9. Find all real x such that $3^x + 3^{x+1} = 324$.

F85J10. In regular nonagon (nine sides) $A_1A_2A_3A_4A_5A_6A_7A_8A_9$ (where the vertices are numbered consecutively), find the degree-measure of angle $A_1A_9A_4$.

Junior Division Contest Number Two — Fall, 1985
Part III Time: 10 Minutes

F85J11. Find the remainder when $7^{31}$ is divided by 4.

F85J12. Circles $O_1$ and $O_2$ have radii 16 and 9 respectively, and are tangent externally at point A. Point B is chosen on circle $O_1$ such that AB = 8, and BC is drawn tangent to circle $O_2$ at point C. Compute the length of BC.

ANSWERS

| | | |
|---|---|---|
| 7. 108 | 9. 4 | 11. 3 |
| 8. $5\sqrt{2}$ | 10. 60 or $60^\circ$ | 12. 10 |

# New York City Interscholastic Mathematics League

Junior Division Contest Number Three — Fall, 1985
Part I Time: 10 Minutes

F85J13. The sum of two numbers is 12, and the difference of their squares is 96. Compute the larger of the two numbers.

F85J14. In right triangle ABC, D is a point on leg BC, M is the midpoint of hypotenuse AB, and MD is perpendicular to AB. If AC = 8 and DC = 6, compute the length of AB.

Junior Division Contest Number Three — Fall, 1985
Part II Time: 10 Minutes

F85J15. Find the smallest positive multiple of 20 which is a perfect cube.

F85J16. In a rectangular coordinate system, there are two circles passing through the point (3,2) and tangent to both coordinate axes. Find the sum of the radii of the two circles.

Junior Division Contest Number Three — Fall, 1985
Part III Time: 10 Minutes

F85J17. The decimal number x97y is divisible by 45. Find all possible ordered (x,y) of decimal digits.

F85J18. In convex hexagon ABCDEF, AB=BC, CD=DE, and EF=FA. If $m\angle ABC + m\angle CDE + m\angle EFA = 360^\circ$, and $m\angle ABC = 100^\circ$, compute the degree-measure of FBD.

## ANSWERS

13. 10

14. $8\sqrt{5}$ or equivalent

15. 1000

16. 10

17. (2,0), (6,5) both required

18. 50 or $50^\circ$

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE

Junior Division    Contest Number One    Fall, 1985

SOLUTIONS

F85J1. Since $31^2 = 961 < 1000 < 1024 = 32^2$, and $70^2 = 4900 < 5000 < 5041 = 71^2$, the squares in the given range are $32^2, 33^2, 34^2, \ldots, 70^2$. There are $70 - 32 + 1 = 39$ of these.

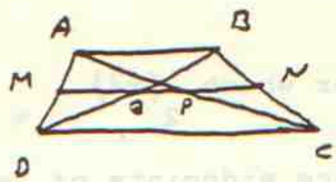
F85J2. $(x+y) - (y-z) = x+z = 34 - 12 = 22$.

F85J3. $AB = \sqrt{6^2 + 4^2} = \sqrt{52} = 2\sqrt{13}$. The area of the triangle is $(1/2)AC \cdot CB = (1/2)AB \cdot CD$. Hence $4 \cdot 6 = 2\sqrt{13} \cdot CD$, and $CD = 12/\sqrt{13}$.

F85J4. If the observer were standing at the far end of the bridge, he would notice the front of the train passing him 30 seconds after it passed the near end of the bridge. Hence the rate of the train is 450/30 = 15 meters per second.

F85J5. $(2^x)^2 = 2^x \cdot 2^x = 2^{2x}$. This must equal $2^{(x^2)}$, so $x^2 = 2x$, and $x = 0$ or 2.

F85J6. Since parallel lines cut off proportional segments on any transversal, Q and P are midpoints of diagonals BD, AC respectively. We now use the theorem that a line connecting the midpoints of two sides of a triangle is half the third side. In triangle ACD, MP = 13/2, while in triangle ABD, MQ = 7/2. Then PQ = MP-MQ = 13/2 - 7/2 = 3.



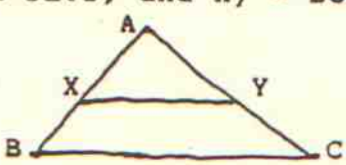
NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE

Junior Division　　　Contest Number Two　　　Fall, 1985

SOLUTIONS

F85J7. We need integers a, b for which $12a = 52b + 4$, or $3a = 13b + 1$, with a as small as possible. A quick trial shows that $a=9$, $b=2$ works, so the number is 108.

F85J8. The ratio of the areas of similar triangles is the square of the ratio of the sides. In triangles ABC, AXY, the ratio of the areas is 2:1, so the ratio of the sides is $\sqrt{2}:1$, and $Xy = BC/\sqrt{2} = 5\sqrt{2}$.

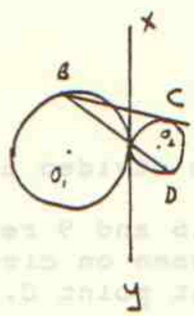



F85J9. We have $3^x + 3^{x+1} = 3^x + 3\cdot 3^x = 4\cdot 3^x = 324$, so $3^x = 81$, and $x = 4$.

F85J10. If we look at the circumscribing circle of the figure, the arcs cut off by each of the sides are $40^\circ$. Hence $m\,\widehat{A_1A_4} = 3\cdot 40 = 120^\circ$, and $m\angle A_1A_9A_4 = (1/2)\cdot 120^\circ = 60^\circ$.

F85J11. Using congruences, $7^2 = 49 \equiv 1 \bmod 4$, so $(7^2)^{15} = 7^{30} \equiv 1 \bmod 4$, and $7^{31} \equiv 7\cdot 1 \equiv 3 \bmod 4$.

F85J12. Let AB intersect circle $O_2$ at D, and let XY be the common tangent to the two circles (see diagram). Then $\angle XAB = \angle YAD$, so $m\,\widehat{AB} = m\,\widehat{AD}$. Hence $AD:AB = 9:16$, and $AD = 9\cdot 8/16 = 9/2$. Then $BC^2 = BD\cdot AB = 8\cdot 25/2 = 100$, so $BC = 10$.



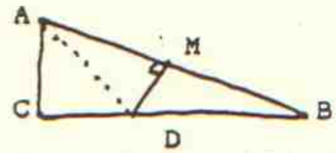
NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE

Junior Division                Contest Number Three                Fall, 1985

SOLUTIONS

F85J13. If the numbers are a and b, then $a+b = 12$ and $a^2-b^2 = (a+b)(a-b) = 96$. Dividing the second equation by the first, we find $a-b = 8$. Solving simultaneously gives $a = 10$, $b = 2$.

F85J14. Draw AD. Applying the Pythagorean theorem to triangle ACD gives AD=10. Since D is on the perpendicular bisector of AB, BD=AD=10. Since BC=16 and AC=8, the Pythagorean theorem shows that $AB = \sqrt{8^2+12^2} = 8\sqrt{5}$ (or equivalent).

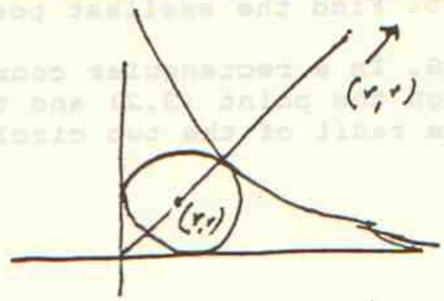



F85J15. In the prime decomposition of a cube, all the exponents must be multiples of 3. Since $20 = 5^1 2^2$, we must multiply it by $5^2 \cdot 2$ to get $5^3 \cdot 2^3 = 1000$ for a perfect cube.

F85J16. Let r be the radius of one of the circles. Then, since the center of such a circle is on the line y=x, we have $(r-3)^2+(r-2)^2 = r^2$, or

$$r^2 - 10r + 13 = 0.$$

We want the sum of the roots of this equation, which is 10.




F85J17.

Since x97y is a multiple of 5, y can only be 0 or 5. Since the number is a multiple of 9, the sum of its digits must be a multiple of 9. If y=0, x must be 2, and if y=5, x must be 6.

F85J18. Let arc AC with center B and arc AE with center F intersect at K (see diagram). Let $m\angle ABC = b^\circ$ (we know here that $b = 100$), $m\angle CDE = d$, and $m\angle BFD = f$. Then $m\overset{\frown}{AC} = b^\circ$, and $m\angle AKC = 180 - (360-b)/2 = 180 - b/2$. Similarly, $m\angle AKE = 180-f/2$. Now $m\angle CKE = 360-m\angle AKC-m\angle AKE = 360-(180-b/2) = b/2 + f/2$. Since $b + d + f = 360^\circ$, $m\angle CKE = 180-d/2$. This measurement matches the measure of an angle inscribed in arc $\overset{\frown}{CE}$ (with center D), so that arc $\overset{\frown}{CE}$ must pass through point K.

Now quadrilaterals ABKE, KBCD are kites, so $\angle KBF = \angle ABF$, $\angle KBD = \angle CBD$, and $m\angle FBD = (1/2)m\angle ABC = b/2$. Here, $m\angle FBD = 50^\circ$.