Spring, 1985 - Senior B - 1 - Questions

S85B1

Express as a common fraction: $1/2-1/3+1/4-1/5+1/6$.

S85B2

In a certain small country, cars have license plate numbers consisting of four-digit numbers. Each digit is either a 1 or a 2. How many different license plates can be issued in this small country?

S85B3

Find all real numbers x for which: $\frac{x^2-1}{x-1}=3$.

S85B4

Find the numerical value of: $\log_2\sqrt{\sqrt{\sqrt{\sqrt{2}}}}$.

S85B5

Jenny has $1.64 in pennies, dimes nickel, and quarters. She has equal numbers of each coin. How many pennies does Jenny have?

S85B6

The second term of a geometric progression of real numbers is 6, and the fifth term is 162. Find the first term.

Answers

1. 23/60
2. 16
3. 2
4. 1/16
5. 4
7. 2

Spring, 1985 - Senior B - 2 - Questions

S85B7

There are 30 natural numbers which divide the number 1200 without remainder. The product of all these 30 numbers can be expressed in the form $12^a \cdot 10^b$. Find the ordered pair of integers (a,b).

S85B8

In triangle ABC, AC = 6, BC = 8, and AB = 10. Points X and Y are chosen between A and B such that AX = AC and BY = BC. Find the length of XY.

S85B9

The roots of the equation $x^2 - px + q = 0$, where p and q are real numbers, are $\sin^2 17°$ and $\cos^2 17°$. Find the numerical value of p.

S85B10

How many natural numbers less than or equal to 200 are divisible neither by 2 nor by 3 nor by 5?

S85B11

A circle is circumscribed about an equilateral triangle. If the area of the circle is 12n square units, we find the number of square units in the area of the triangle.

S85B12

When the polynomial $x^3 - x^2 + 5k - 2$ is divided by x – 3, the remainder is k. Find the numerical value of k.

Answers

7. (15,35)
8. 4
9. 1
10. 54
11. $9\sqrt{3}$ or equivalent
12. –4

Spring, 1985 - Senior B - 3 - Questions

S85B13

Two grand masters play a series of gams of chess. At the end of each game, the loser must pay the winner one dollar. At the end of the series, Boris has won 4 games, while Bobby has won 4 dollars. If none of the games ended in a stalemate (a tie), how many games did they play in all?

S85B14

In the figure, secants $\overparen{PR}$ and $\overparen{PS}$ are drawn to the circle. The degree-Measure of angle P is 30 and the degree-measure of angle RXS is 70. What is the degree-measure of arc $\overparen{RS}$?

S85B15

Find the numerical value of $\log_9\left(\sqrt{3}/27\right)$.

S85B16

Find the (positive) difference between the two roots of the equation $x^2 + 9x + 12 = 0$.

S85B17

Find the degree-measure of the acute angle formed by the hands of a clock when the clock shows 5:15 exactly.

S85B18

The real number $\tan 15°$ can be expressed in the form $a + b\sqrt{3}$, where a and b are integers. Find the ordered pair (a,b).

Answers

13. 8
14. 100 or 100°
15. –1.25
16. $\sqrt{33}$ or equivalent
17. 67.5 or equivalent
18. (2,-1)

Spring, 1985 - Senior B - 4 - Questions

S85B19

Three boxes are numbered 1, 2, and 3. Inside each box is a slip of paper with the number 1, 2, or 3 on it. The number on the slip in the box marked n is the number of the box containing the slip marked n. If the number 1 contains tha slip marked 3, what number is on the slip in box 2?

S85B20

If $\sin A = \frac{3}{5}$ and cos A is positive, find the numerical value of sin 2A.

S85B21

Find a three-digit integer multiple of 45 (in base 10 notation) whose digits, taken one at a time from left to right, form an arithmetic progression with positive common difference.

S85B22

Find all real numbers x such that $\log(x+1)+\log(x-1)=\log(x^2-x-1)$, where the base of all logarithms is 10.

S85B23

In the figure, a circle is drawn tangent to each of the sides of triangle ABC at points P, Q, and R. If AB = 11, AC = 6, and BC = 7, find the length of BP.

S85B24

One root of the equation $x^2 - px + 24 = 0$ is three times the other root. If both roots are positive, find the numerical value of p.

Answers

19. 2
20. 24/25
21. 135
22. 2
23. 6
24. $8\sqrt{2}$ or equivalent

Spring, 1985 - Senior B - 5 - Questions

S85B25

Find the largest odd integer which is a multiple of 5, and which, when written in base 10 notation, contains no two digits which are the same.

S85B26

Find the sum of the 5 angles at the points of the star at right.

S85B27

The number $3^{12} - 3$ is written in base 9 notation. Find the two last (rightmost) digits.

S85B28

The pages of a fat Russian novel are numbered consecutively, starting with page one. The novel is bound in three volumes, with an equal number of pages in each volume. The sum of the numbers on the first page of each volume is 1353. How many pages are there in each volume?

S85B29

Tom has a set of k natural numbers. No two of Tom's numbers differ by a multiple of 10. At most, how many numbers could be in Tom's set?

S85B30

Find the smallest positive integer n such that n + 125 and n + 201 are both perfect squares.

Answers

25. 9876432105
26. 180 or 180°
27. 86
28. 450
29. 10
30. 199

Spring, 1985 - Senior B - 1 - Solutions

S85B1

The least common denominator is 60, and: $30/60-20/60+15/60-12/60+10/60=55/60-32/60=23/60$.

S85B2

In making up a license number, we must choose between the digits 1 and 2 four times. This makes $2\cdot 2\cdot 2\cdot 2=16$ possible license plates.

S85B3

Clearly $x\neq 1$. For any other possible value of x, $\dfrac{x^2-1}{x-1}=x+1$, and $x+1=3$ only for $x=2$.

S85B4

We have:

$$\begin{aligned}\log_2\sqrt{\sqrt{\sqrt{\sqrt{2}}}} &= \frac{1}{2}\log_2\sqrt{\sqrt{\sqrt{2}}}=\frac{1}{4}\log_2\sqrt{\sqrt{2}}\\ &=\frac{1}{8}\log_2\sqrt{2}=\frac{1}{16}\log_2 2\\ &=\frac{1}{16}\end{aligned}$$

S85B5

Method I:

If Jenny has x of each coin, then 25x + 10x + 5x + x = 41x = 164, so x = 4.

Method II:

Jenny must have at least 4 pennies, since she has $1.64 altogether. If she had more pennies, she would have at least 9. But then she would have 9 quarters as well, which is too much. Hence she has exactly four pennies, so she has four of each coin.

S85B6

If a is the first term, and r the common ratio, then ar = 6 and $ar^4=162$. Dividing, we find that $r^3=27$, so r = 3. Hence a = 2.

Spring, 1985 - Senior B - 2 - Solutions

S85B7

If n is a divisor of 1200, then so is 1200/2. Since 1200 is not a perfect square, these two divisors cannot be equal. Therefor the divisors of 1200 come in pairs whose product is n. There are 15 of these pairs, so the product of all the divisors is $1200^{15} = 12^{15} \cdot 10^{15}$.

S85B8

We have AX + BY = 8 + 6 = 14 = AX + XY + BX + XY = AB + XY = 10 + XY, so
XY = 4.

S85B9

The sum of the roots of the equation $x^2 - px + q = 0$ is p. Since $\sin^2 A + \cos^2 A = 1$ for any angle A, $p = \sin^2 17° + \cos^2 17° = 1$.

S85B10

If A, B, and C are three sets, and we use absolute value to denote the number of elements in a set, then: $|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$. We can use this to count the numbers we want if take the universe to be natural numbers less than or equal to 200, and let A = {multiples of 2}, B = {multiples of 3}, C = {multiples of 5}. Then A∩B = {multiples of 6}, B∩C = {multiples of 15}, A∩C = {multiples of 10}, and A∩B∩C = {multiples of 30}. We have:

$$
\begin{aligned}
|A| &= [200/2] = 100 \\
|B| &= [200/3] = 66 \\
|C| &= [200/5] = 40 \\
|A \cap B| &= [200/6] = 33 \\
|B \cap C| &= [200/15] = 13 \\
|A \cap C| &= [200/10] = 20 \\
|A \cap B \cap C| &= [200/30] = 6 \\
|A \cup B \cup C| &= 206 - 66 + 6 = 146
\end{aligned}
$$

Brackets denote the “greatest integer” function.

This tells how many numbers are divisible either by 2 or by 3 or by 5. The complement of this set is the set of numbers divisible by neither 2 nor 3 nor 5. There are 200 – 146 = 54 of these.

S85B11

The radius of the circle is $\sqrt{12} = 2\sqrt{3}$. From triangle AMX (see diagram), we have AX = 3, so AB = 6. The area of an equilateral triangle of side 6 is $9\sqrt{3}$.

S85B12

Let $f(x) = x^3 - x^2 + 5k - 2$. Then the remainder upon division by x – 3 is (by the remainder theorem) $f(3) = 16 + 5k$. Hence $6 + 5k = k$, and k = -4.

Spring, 1985 - Senior B - 3 - Solutions

S85B13

Since Bobby won four games, and Boris also won four games, they played 8 games altogether.

S85B14

Let $m\widehat{RS} = x$, $m\widehat{QP} = y$. Then:

$$mP = 30 = \frac{1}{2}\left(x - y\right)$$

$$mRXQ = 70 = \frac{1}{2}\left(x + y\right)$$

Adding, we find 100 = x.

S85B15

$$\frac{\sqrt{3}}{27} = \frac{3^{0.5}}{3^3} = 3^{-2.5} = \left(9^{0.5}\right)^{-2.5} = 9^{-1.25}$$

Hence $\log_9 \frac{\sqrt{3}}{27} = -1.25$.

S85B16

Using the quadratic formula, we find that the positive difference between the roots of the equation $ax^2 + bc + c = 0$ is $\frac{\sqrt{b^2 - 4ac}}{a}$. In the present case, this is $\sqrt{81 - 48} = \sqrt{33}$.

S85B17

At 5:00, the hands make an angle of $150°$. If the hour hand did not move from its 5:00 position, the angle at 5:15 would be $60°$. But in fifteen minutes, the hour hand moves away from the minute hand by and angle of $\frac{1}{4}\left(30°\right) = 7.5°$. Hence the angle formed by the hands at 5:15 is actually $67.5°$.

S85B18

We start with the formula: $\tan\left(\frac{x}{2}\right) = \pm\sqrt{\frac{1 - \cos\alpha}{1 + \cos\alpha}}$. Letting $x = 30°$, $\cos x = \frac{\sqrt{3}}{2}$, so: $\tan 15 = \sqrt{\frac{1 - \frac{\sqrt{3}}{2}}{1 + \frac{\sqrt{3}}{2}}} = \sqrt{\frac{2 - \sqrt{3}}{2 + \sqrt{3}}} = \sqrt{7 - 4\sqrt{3}}$. We must express this number in the required form. Let $\left(a + b\sqrt{3}\right)^2 = 7 - 4\sqrt{3}$. Then $a^2 + 3b^2 = 7$ and $2ab = -4$. This is satisfied for (a,b) = (-2,1) or (2,-1). The first ordered pair gives us a negative answer, so we must reject it.

Spring, 1985 - Senior B - 4 - Solutions

S85B19

Since box 1 contains slip 3, box 3 must contain slip 1. This leaves slip 2, which must then be in box 2, and the conditions of the problem are satisfied.

S85B20

Since $\cos^2 A + \sin^2 A = 1$, $\cos^2 A = 16/25$, and $\cos A = 4/5$. Then $\sin 2A = 2(\sin A)(\cos A) = 2(3/5)(4/5) = 24/25$.

S85B21

Since the number we want is a multiple of 45, it must also be a multiple of 5, so its units digit is 0 or 5. An arithmetic progression formed from the digits 0 through 9 (with positive common difference) cannot have its least term 0, so the units digit is 5. Only the numbers 135 and 345 are left to try out, and only the first is a multiple of 45.

S85B22

We have:

$$(x+1)(x-1) = x^2 - x + 1$$
$$-1 = -x + 1$$
$$2 = x$$

S85B23

Let BP = BR = x, CP = CQ = y, AR = AP = z. Then:

$$x + y = 7$$
$$y + z = 6$$
$$x + z = 11$$

Adding, 2x + 2y + 2z = 24, and x + y + z = 12. Subtracting the second equation from this, x = 6.

S85B24

We can represent the roots by r and 3r. Using the formula for the product of the roots $3r^2 = 24$, and $r = 2\sqrt{2}$ (since the roots are positive). Hence the second root is $6\sqrt{2}$. Their sum is $8\sqrt{2}$, which is p.

Spring, 1985 - Senior B - 5 - Solutions

S85B25

We must create the required integer from one copy each of the digits 0 through 9. We must use 5 as the units digit to obtain an odd multiple of 5. We must make the largest igit have the largest place-value to obtain the largest possible number. These conditions are satisfied by the number 9876432105.

S85B26

Letting x and y represent the measures of the angles shown in the diagram, we have:

(from triangle ACP) $A + C + x = 180$

(from triangle BEQ) $B + E + y = 180$

(from triangle DPQ) $D + (180 - x) + (180 - y) = 180$.

Adding, we find $A + B + C + D + E + 2 \cdot 180 = 3 \cdot 180$. Hence $A + B + C + D + E = 180$.

S85B27

Working in decimal notation, we can write $3^{12} - 3 = 9^6 - 3$. Switching to base nine notation, this number can be written as $1{,}000{,}000 - 3$. Using the usual subtraction algorithm, but "borrowing" nines rather than tens, we can write this as 888,886.

S85B28

If each volume as n pages, then:

$$\begin{aligned} 1 + (n+1) + (2n+1) &= 1353 \\ 3n + 3 &= 1353 \\ n &= 450 \end{aligned}$$

S85B29

Tom can certainly have ten numbers (for example, the numbers from 1 to 10). But if he has eleven numbers, then two of them will have the same remainder when divided by 10 (since only ten remainders are possible). The difference between two such numbers will be a multiple of ten.

S85B30

This problem concerns the size of the spaces between perfect squares. Since (n + 201) – (n + 125) = 76, we are looking for two perfect squares which differ by 76. Representing these as a^2 and b^2, we have:

$$\begin{aligned} a^2 - b^2 &= 76 \\ (a+b)(a-b) &= 76 \end{aligned}$$

Since both factors on the left are integers, we can solve for a and b by factoring 76:

$$76 = 76 \cdot 1$$

$$76 = 38 \cdot 2$$

$$76 = 19 \cdot 4$$

If $a + b = 76$ and $a – b = 1$, then $2a = 77$, and a is not an integer. The same thing happens if we take the third pair of factors above. But if we let $a + b = 38$, $a – b = 2$, we find that $a = 20$ and $b = 18$. This leads to $n = 199$, which is the unique solution for positive integers.