# Spring, 1985 - Senior A - 1 - Questions

### *S85S1*

Numbers a and b can be represented by 2-digit decimal numbers, one having the same digits as the other in reverse order. The difference a-b has the digit 5. What is the other digit?

### *S85S2*

A sequence $x_n$ has the property that $x_1 = x_{1000} = 0$ and $x_n = \dfrac{x_{n+1} + 2x_{n-1}}{3}$. Find $x_{500}$.

### *S85S3*

Find all values of $\theta$ between $0°$ and $180°$ such that $(\csc\theta + \cot\theta)(1 - \cos\theta) = \cos 23°$.

### *S85S4*

Point O is the center of a circle circumscribing triangle ABC. A second circle, passing through points A, B and O is tangent to line BC. If $m\angle ABC = 20°$, find the degree-measure of $\angle ABC$.

### *S85S5*

A father is as old in years as his son was in months when the father was 9 times as old as the son. The father is now 26 years and 8 months older than the son. How many years old is the father?

### *S85S6*

Let $f(x)$ be a real polynomial in $x$ such that $f(x^3) = x^4 f(x)$ for all $x$, and $f(1) = 2$. Find $f(2)$.

### *Answers*

1. 4
2. 0
3. $67°, 113°$; both required
4. 80 or $80°$
5. 40
6. 8

## Spring, 1985 - Senior A - 2 - Questions

### *S85S7*

Let E and F be the midpoints of sides $\overline{AB}$ and $\overline{CD}$ respectively of the square ABCD, which has sides of unit length. Denote by G the intersection of line $\overline{DE}$ and diagonal $\overline{AC}$, and let H be the intersection of line $\overline{BF}$ and $\overline{AC}$. Calculate the length of $\overline{GH}$.

### *S85S8*

Find the smallest positive value of $m$ such that the solutions (for $x$) of $\frac{x^2+x+1}{x} = m$ are real numbers.

### *S85S9*

The first term of a geometric progression is $i$, and the fifth term is $4+3i$. The third term is $a+bi$, where $a$ and $b$ are real numbers and $a>0$. Find the ordered pair ($a$,$b$).

### *S85S10*

For all real numbers $x$, $f(x) = \sqrt{x^2+25} + \sqrt{(x-1)^2+25}$. If the minimum value of $f(x)$ is $\sqrt{a}$, find $a$.

### *S85S11*

A line whose equation is $y = mx + b$ passes through the point (1,1) and intersects the line $x - 5y + 23 = 0$ at point A. It intersects the line $x - 5y + 11 = 0$ at point B. The midpoint of segment AB lies on the line $2x - y = 2$. Find the ordered pair ($m$,$b$).

### *S85S12*

A sequence $< a_i >$ of integers in decimal notation is formed as follows: $a_1 = 2$, $a_i =$ the last integer such that $a_i > a_{i-1}$ <u>and</u> the product of the digits of $a_i$ is prime (1 is not prime). If $a_i$ has but one digit we take $a_i$ itself to be the product of its digits. Evaluate $a_{61} - a_{60}$.

### *Answers*

7. $\frac{\sqrt{2}}{3}$
8. 3
9. (1.2)
10. 101
11. (1.5,-.5) or equivalent ordered pair
12. 40,001

## Spring, 1985 - Senior A - 3 - Questions

### *S85S13*

Two sides of an isosceles triangle are tangent to a semicircle of radius 10, and the base of the triangle contains the diameter of the semicircle. Find the smallest possible area for the triangle.

### *S85S14*

Find the units digit in the base 10 decimal representation of $3^{3^{3^{3}}} = 3^{\left(3^{\left(3^{3}\right)}\right)}$.

### *S85S15*

Point F is chosen inside regular pentagon ABCDE so that triangle CDF is equilateral. Find the degree-measure of angle BFE.

### *S85S16*

The positive integers $p$ and $q$ are both less than 10. How many quadratic equations of the form $x^2 - px + q = 0$ have positive integers for all their roots?

### *S85S17*

Find all real numbers $x$ such that $\sqrt{3+\sqrt{x}} + \sqrt{4-\sqrt{x}} = \sqrt{7}$.

### *S85S18*

For all right triangles with hypotenuse of length $k$, the largest possible value of the radius of the inscribed circle can be expressed explicitly in terms of $k$ as $k\left(a+b\sqrt{2}\right)$. Find the ordered pair ($a$,$b$) of rational numbers.

### *Answers*

13. 200
14. 7
15. 168 or 168°
16. 13
17. 16

13. (-.5,.5) or equivalent

## Spring, 1985 - Senior A - 4 - Questions

### *S85S19*

The finite sequence $S = < a_0, a_1, a_2, ..., a_n >$ is constructed in such a way that the number $a_i$ is the number of elements of S which are equal to i. Find the sum $a_0 + a_1 + a_2 + ... + a_9$.

### *S85S20*

Find all (positive) prime numbers which are 27 less than the cube of an integer.

### *S85S21*

If each angle of triangle ABC is no smaller than $50°$, find the degree measure of the largest possible angle the triangle can contain.

### *S85S22*

A function is defined on the set of positive integers as follows: $f(1) = 1$ and $f(n+1) = f(n) + 2\sqrt{f(n)} + 1$ for all odd n; $f(n+1) = f(n) + 2\sqrt{f(n) + 2} + 3$ for all even n.
Find $f(101)$.

### *S85S23*

Line segments OA, OB, OC are non-coplanar. Points A’ and B’ are chosen on segments AO and BO respectively such that A’O:AO = 1:2 and B’O:BO = 1:3. Point C’ is chosen on ray $\overrightarrow{OC}$ so that the volumes of tetrahedron OABC, OA’B’C’ are equal. Find the numerical value of the ratio OC’:OC.

### *S85S24*

In the right triangle ABC, CM is the median to hypotenuse AB, and CT is an angle bisector. If CT = 2 and $CM = \sqrt{12}$, find the area of triangle ABC.

### *Answers*

19. 10
20. 37
21. 62 or $62°$
22. 10201
23. 6 or 6:1
13. 6

## Spring, 1985 - Senior A - 5 - Questions

### *S85S25*

Find the units digit in the decimal representation of the integer $3^{4^{5^6}} = 3^{\left(4^{\left(5^6\right)}\right)}$.

### *S85S26*

A chef at a summer camp knows how to prepare only two dishes. He prepares only one dish each day. The camp does not allow him to cook the same dish more than twice in a row (on two successive days). How many different 10-day menus can he prepare?

### *S85S27*

How many two-digit numbers (in decimal notation) are divisible by the product of their digits?

### *S85S28*

A circle with the center S has radius 1. Triangle ABC is circumscribed about the circle, And $SA \leq SB \leq SC$. Find the area of the region of the plane in which point A may be located.

### *S85S29*

Find two natural numbers, both greater than $2^{500}$, whose product is $2^{2002}+1$.

### *S85S30*

Line segment AB has length 4 units. Two arcs, with centers at A and B and with radius 4, intersect at D. Point E is the midpoint of AB, and semicircles with diameters AE and BE are drawn on the same side of line AB as D. Point M is the center of a circle tangent to these two semicircles and also of arcs AD and BD. Find the length of EM.

### *Answers*

25. 1
26. 178
27. 5
28. $3\pi$
29. $2^{1001}+2^{501}+1$
    $2^{1001}-2^{501}+1$
    (both required)
30. $4\sqrt{6}/5$
    or equivalent

## Spring, 1985 - Senior A - 1 - Solutions

### *S85S1*

The numbers can be represented by $10t+u$ and $10u+t$. The difference is then $9t-9u=9(t-u)$, and is a multiple of 9. The only suitable multiples of 9 are 45 and 54. In either case, the other digit is 4.

### *S85S2*

We can write $x_{n+1}=3x_n-2x_{n-1}$. From this recursion relation it is clear that the values of $x_1$ and $x_2$ determine all the values of the sequence. If we let $x_2=2$, we find $x_3=3a$, $x_4=7a$, $x_5=15a$ and an easy induction shows that $x_n=\left(2^{n-1}-1\right)a$. If $x_2\neq 0$, $x_n$ cannot be 0. Hence $x_2=0$, and the sequence is constantly 0.

### *S85S3*

$$(\csc\theta+\cot\theta)(1-\cos\theta)=\left(\frac{1+\cos\theta}{\sin\theta}\right)(1-\cos\theta)=\frac{1-\cos^2\theta}{\sin\theta}=\frac{\sin^2\theta}{\sin\theta}=\sin\theta=\cos 23^\circ.$$

Therefore $\theta=67^\circ,113^\circ$. (Note that $\sin\theta$ is positive between $0^\circ$ and $180^\circ$.)

### *S85S4*

In the second circle, $\angle BAO$ is inscribed and cuts off arc $\overset{\frown}{BO}$. Angle OBC is a tangent-chord angle intercepting the same arc. Hence $\angle BAO\cong\angle OBC$. Since OA=OB=OC, triangles ABO and CBO are thus congruent, so AB=BC. If $m\angle ABC=20$, $m\angle ABC=\frac{1}{2}160^\circ=80^\circ$.

### *S85S5*

The father has always been $26\frac{2}{3}=\frac{80}{3}$ years older than the son. When the son was x years old, the father was 9x years old. Hence $8x=\frac{80}{3}$ and $x=\frac{10}{3}$. The father is now 12x years old, or 40.

### *S85S6*

First let us find the degree of $f$. If $\deg(f)=n$, we have $\deg f\left(x^3\right)=3n$ and $\deg\left[x^4 f(x)\right]=n+4$, so $3n=n+4$ and $n=2$. Note that $f(0)=0$, so we can write $f(x)=ax^2+bx$. Then $f\left(x^3\right)=ax^6+bx^5$, and $b=c$. Since $f(1)=2$, $f(x)=2x^2$, and $f(2)=8$.

## Spring, 1985 - Senior A - 2 - Solutions

### *S85S7*

By similar triangles: $\frac{CF}{FD} = \frac{CH}{HG}$ and $\frac{AE}{EB} = \frac{AG}{GH}$. Since AE = EB and *CF* = *FD*, the quantities *AG*, *GH*, and *HC* must be equal. Therefor $GH = \frac{1}{3}AC = \frac{\sqrt{2}}{3}$.

### *S85S8*

We have $x + 1 + \frac{1}{x} = m$, $x + \frac{1}{x} = m - 1$. Since $m > 0$, and real solution for $x$ will also be greater than 0, we have for $x > 0$, $x + \frac{1}{x} \geq 2$. Hence $m \geq 3$.

### *S85S9*

$(a + bi)^2 = a^2 \quad b^2 + 2abi = i(4 + 3i) = -3 + 4i$.

Then $a^2 - b^2 = -3$ and $2ab = 4$ or $ab = 2$. Trying $a = 1$, $b = 2$ works. One can also show that this solution is unique.

### *S85S10*

Solution 1:

$\frac{p+q}{2} \geq \sqrt{pq}$ if $p \geq 0$ and $q \geq 0 \Rightarrow$

$f(x) = \sqrt{x^2 + 25} + \sqrt{(x-1)^2 + 25} \geq 2\sqrt{(x^2 + 25)[(x-1)^2 + 25]}$ and of course we have equality when $p = q$, i.e. when $\sqrt{x^2 + 25} = \sqrt{(x-1)^2 + 25}$, i.e. when $x^2 = (x-1)^2$, i.e. when $x = -(x-1)$, i.e. when $x = \frac{1}{2} \Rightarrow$ minimum value of

$f(x) = f\left(\frac{1}{2}\right) = \sqrt{\frac{1}{4} + 25 +} + \sqrt{\frac{1}{4} + 25} = 2\sqrt{\frac{1}{4} + \frac{100}{4}} = \sqrt{101}$. a = 101.

Solution 2: (Using a geometric interpretation)

$f(x) = \left|\overline{AX}\right| + \left|\overline{XB}\right|$ we get minimum by considering line *AXC*.

### *S85S11*

A segment with endpoints on two parallel lines has its midpoint on a third line midway between them. Hence the midpoint of AB is on the line $x - 5y + 17 = 0$. Solving simultaneous equations $\begin{array}{l} x - 5y + 17 = 0 \\ 2x - y - 2 = 0 \end{array}$, we find the coordinates of this midpoint to be (3,4). The line through this point and also (1,1) has equation $y = 1.5x - .5$.

***S85S12***

The product of the digits can only be 2,3,5 or 7. In fact, each $a_i$ must consist of <u>one</u> of these digits and possibly some digits equal to 1.

| | | | |
|---|---|---|---|
| There are 4 | 1-digit | $a_i$ | (2,3,5,7) |
| 8 | 2-digit | $a_i$ | (12,13,…,51,71) |
| 12 | 3-digit | $a_i$ | (112,…,711) |
| 16 | 4-digit | $a_i$ | (1112,…,7111) |
| <u>20</u> | 5-digit | $a_i$ | (11112,…,71111) |
| 60 | | | |

Therefor $a_{60} = 71{,}111$ so $a_{61} = 111{,}112$ and $a_{61} - a_{60} = 40{,}001$.

## Spring, 1985 - Senior A - 3 - Solutions

### *S85S13*

If the triangle is $ABC$ (see diagram) with $AC = BC$, then $OP = OQ = 10$, and $|ABC| = |AOC| + |BOC| = \frac{1}{2}10(AC + BC) = 10AC$. This is minimal when $AC$ is minimal. Let $\angle CAO = \theta$. Then $AP/OP = \cot\theta$, while $CP/OP = \tan\theta$.
$AC = AP + CP = OP\cot\theta + OP\tan\theta = 10(\cot\theta + \tan\theta)$. Since $(\tan\theta)(\cot\theta) = 1$, a constant, the sum of these to quantities is minimal when they are equal. This makes $\theta = 45°$, and the areas of the right isosceles triangle is 200.

### *S85S14*

$3^2$ ends in 9 $\Rightarrow$ $3^4$ ends in 1 $\Rightarrow$ $3^{4n}$ ends in 1 for all integers $n \ge 0$. Now,
$3^{3^{3^3}} = 3^{3^{27}} = 3^{(4-1)^{27}} = 3^{4^{27} - 4^{26}\binom{27}{1} + 4^{25}\binom{27}{2} - \ldots + 4^{26}\binom{27}{26} - 1} = 3^{4n-1}$ (for some integer $n > 0$). $\Rightarrow$
$3^{3^{3^3}} = 3^{4(n-1)+3}$, $3^{4(n-1)}$ ends in 1, $3^3$ ends in 7 $\Rightarrow$ $3^{3^{3^3}}$ ends in 7.

### *S85S15*

We have: $m\angle BCF = m\angle BCD - m\angle FCD = 108 - 60 = 48$, so
$m\angle BCF = \frac{1}{2}(180 - 48) = 66$, and $m\angle ABF = 360 - (66 + 66 + 60) = 360 - 192 = 168$.

### *S85S16*

If the quadratic equation $x^2 - px + q = 0$ has positive integral roots, then $x^2 - px + q$ factors into $(x - a)(x - b)$, where $a$ and $b$ are positive integers whose sum is $p$ and whose product is $q$. We can count the equations easily by giving $q$ a value, finding pairs of factors for $q$, and noting their sums.

For q = 1 we have $x^2 - 2x + 1$.
For q = 2 we have $x^2 - 3x + 2$.
For q = 3 we have $x^2 - 4x + 3$.
For q = 4 we have $x^2 - 5x + 4$ and
$x^2 - 4x + 4$.
For q = 5 we have $x^2 - 6x + 5$.
For q = 6 we have $x^2 - 7x + 6$ and
$x^2 - 5x + 6$.
For q = 7 we have $x^2 - 8x + 7$.
For q = 8 we have $x^2 - 9x + 8$ and
$x^2 - 6x + 8$.
For q = 9 we have $x^2 - 10x + 9$ and
$x^2 - 6x + 9$.
This makes 13 ordered pairs in all.

Can you generalize this pattern?

### *S85S17*

If $\sqrt{a}+\sqrt{b}=\sqrt{a+b}$, then squaring we find $ab+2\sqrt{ab}=a+b$ and $a=b=0$. Here, $3+\sqrt{x}+4-\sqrt{x}=7$, so we must have $\sqrt{x}=-3$ or $\sqrt{x}=4$, and $x$ = 9 or 16. Checking, we find only $x$ = 16 works.

### *S85S18*

We apply the following theorem.

Thm. For any triangle $k = rs$, where $k$ is the area of the triangle, $r$ *is* the inradius, and $s$ is the triangles semiperimeter.

Corollary: If $a$, $b$, and $c$ are the sides of a right triangle where $c$ is the hypotenuse, then the inradius $r=\dfrac{a+b-c}{2}$.

If $\theta$ is an acute angle of our given triangle then the other two sides are $k\sin\theta$ and $k\cos\theta$, from which it follows that $r=\dfrac{k\sin\theta+k\cos\theta-k}{2}$.

We need to maximize $k\sin\theta+k\cos\theta\geq 2\sqrt{k^2\sin\theta\cos\theta}=2k\sqrt{\dfrac{\sin 2\theta}{2}}$, which is clearly maximized when $\theta=45°$. Thus the maximum value of $r$ is $\dfrac{k\frac{\sqrt{2}}{2}+k\frac{\sqrt{2}}{2}-k}{2}=\dfrac{k\left(\sqrt{2}-1\right)}{2}$.

## Spring, 1985 - Senior A - 4 - Solutions

### *S85S19*

Since there are 10 elements altogether of the sequence, the sum of the $a_i$'s must be 10.

### *S85S20*

If p is such a prime then for some integer, x, we have that $p = x^3 - 27 = x^3 - 3^3 = (x-3)(x^2+3x+9) \Rightarrow x - 3 = 1, -1, p, -p$. If x = p + 3, $x^2 + 3x + 9 = 1$, so p cannot be prime. If x = p – 3, $x^2 + 3x + 9 = p^2 - 3p + 9$ cannot be equal $\pm 1$, so p cannot be prime. A quick check shows that we get a solution if and only if x – 3 = 1 ⇒ x = 4 ⇒ $p = 4^3 - 27 = 37$.

### *S85S21*

Suppose $\angle A$ is the largest (with no loss of generality). Then
$\angle A = 180 - (\angle B + \angle C) < 180 - (2 \cdot 59) = 62°$.

### *S85S22*

Let us relate $f(n)$ and $f(n+2)$ if n is odd. Since n + 1 is even, we have:

$$f(n+1) = f(n) + 2\sqrt{f(n)} + 1$$

$$f(n+2) = f(n+1) + 2\sqrt{f(n+1)} - 2 + 3 = f(n) + 2\sqrt{f(n)} + 1 + 2\sqrt{f(n) + 2\sqrt{f(n)} + 1} + 1$$

$$= \left[\sqrt{f(n)} + 1\right]^2 + 2\sqrt{\left[\sqrt{f(n)} + 1\right]^2} + 1$$

$$= \left[\sqrt{f(n)} + 1\right]^2 + 2\left[\sqrt{f(n)} + 1\right] + 1$$

$$= \left[\sqrt{f(n)} + 1 + 1\right]^2 = \left[\sqrt{f(n)} + 2\right]^2$$

### *S85S23*

We use absolute value for both area and volume. Looking at plane ABO, we see that $\dfrac{|ABO|}{|A'B'O|} = \dfrac{\frac{1}{2} OA \cdot OB \sin A}{\frac{1}{2} OA' \cdot OB' \sin A} = \dfrac{OA}{OA'} \cdot \dfrac{OB}{OB'} = \dfrac{6}{1}$.

Now we look on the plane through OC perpendicular to plane OAB. The ratio $\dfrac{OC}{OC'} = \dfrac{OP}{OP'}$ where P and P' are the feet of the perpendiculars from C and C' respectively to plane OAB. If the tetrahedra OABC, OA'B'C' are equal in volume, and the bases OA'B', OAB are in the ratio 1:6 the heights CP', CP are in the ratio 6:1.

### *S85S24*

Let AC = x, BC = y. We want to find xy/2, so we will look for relationships between x and y.

Since CM = AM = BM, the Pythagorean theorem shows that $x^2 + y^2 = 48$. We can get a second relation by using the formula for the area of a triangle in terms of two sides and the sine of the included angle. Using absolute value for area, we have: $|ABC| = |ACT| + |BCT|$, or $xy/2 = x/\sqrt{2} + y/\sqrt{2}$, so that $xy = \sqrt{2}(x + y)$. Squaring, $x^2y^2 = 2(x^2 + y^2) + 4xy = 96 + 4xy$. Now we let xy/2 = k, the area we want to find. We then have:

$$4k^2 - 8k - 96 = 0$$

$k^2 - 2k - 24 = 0$ We reject the negative root.

$$(k - 6)(k + 4) = 0$$

## Spring, 1985 - Senior A - 5 - Solutions

### *S85S25*

Since $3^4 = 81$ ends in a 1, $3^n$ ends in a 1 whenever n is multiple of 4. This is clearly the present case.

### *S85S26*

It is not hard to see that half of all the possible menus begin with one of the dishes on the first day, and the other half with the other dish. Thus we can assume that the first dish is “fixed”, count the menus, then multiply by two. Let $f(n)$ represent the number of n-day menus the chef can plan, starting with one of the dishes the first day. Clearly $f(1) = 1$ and $f(2) = 2$. We will express $f(n+1)$ in terms of $f(n)$ and $f(n-1)$.
The value of $f(n+1)$ is no less than $f(n)$, since the chef can always get a permissible menu by making the dish for the (n – 1)st day different from that of the day before. He has an extra choice, for the (n + 1)st dish, for each time he changed the dish in going from the (n – 1)st to the nth day. He can make such a change exactly $f(n-1)$ times. Thus $f(n+1) = f(n) + f(n-1)$. A quick calculation shows that $f(10) = 89$, so he has 178 menus in all.
This Fibonacci recursion can be proved formally by induction. The diagram below provides motivation:

### *S85S27*

If the number is 10t + u, and tu divides it, than 10t + u is a multiple of both t and u. Hence t divides u, and u divides 10t. Let u = kt. Then kt divides 10t, so k divides 10, or k = 1,2,5. For k = 1 we have 10t + u = 11. For k = 2 we have 12, 24, 36. For k = 5 we have 15, for five solutions in all.

### *S85S28*

Intuitively, it is not hard to see A must be closer than a vertex of the equilateral triangle circumscribed about circle S. Using this insight, we see that the locus for A is an annulus of inner radius 1 and outer radius 2.Its area is $\pi(R^2 - r^2) = 3\pi$.

Let us prove this result. Suppose triangle ABC is a triangle circumscribed about the circle, with angles $BAC = \alpha$, $CBA = \beta$, $BCA = \chi$. Then, from right triangle SAX, $\sin\alpha/2 = 1/SA$. Similarly, $\sin\beta/2 = 1/SB$ and $\sin\chi/2 = 1/SC$. Thus the condition $SA \le SB \le SC$ means that $\sin\alpha/2 \ge \sin\beta/2 \ge \sin\chi/2$, or (since the three half-angles are all acute), $\alpha/2 \ge \beta/2 \ge \chi/2$, and $\alpha \ge \beta \ge \chi$. Hence $\alpha$ is at least 60 degrees, and $SA = 1/\sin\alpha/2 \le 2$.
This shows that the point A must lie in the annulus. It is not hard to see that any point in the annulus in fact satisfies the conditions of the problem.

***S85S29***

We have

$$\begin{aligned} 2^{2002}+1 &= 2^{2002}+2^{1002}+1-2^{1002} \\ &= 2^{2002}+2\cdot 2^{1001}+1-2^{1002} \\ &= \left(2^{1001}+1\right)^2-\left(2^{501}\right)^2 \\ &= \left(2^{1001}+2^{501}+1\right)\left(2^{1001}-2^{501}+1\right) \end{aligned}$$

To show that the second factor is large enough:

$2^{1001}-2^{501}+1>2^{1001}-2^{501}=2\left(2^{1000}-2^{500}\right)>2\cdot 2^{500}>2^{500}$.

Other solutions may be possible.

***S85S30***

The common centerline of two tangent circles passes through their point of contact. Hence AM intersects DB at K, the point of contact of circle M and arc DB. Similarly, if F is the midpoint of AE, MF passes through X, the point of contact of circle M and semicircle AB.

Let x = MK, y = ME. We will use right triangles AME, FME to solve for x and y. We have:

$$AM = AK - KM = 4 - x$$

$$AE = 2$$

$$MF = MX + XF = x + 1$$

$$(4-x)^2 = y^2 + 4$$

$$(1+x)^2 = y^2 + 1$$

Subtracting, we find –10x + 5 = 3, and x = 6/5. It quickly follows that $y = 4\sqrt{6}/5$.