# New York City Interscholastic Mathematics League

Senior A Division Contest Number 1 Fall, 1984

Part I Time: 11 minutes NYCIML - Sr. A Contest 1 Fall, 1984

F84S1. In square ABCD, points P,Q,R, and S are chosen respectively on sides AB,BC,CD,DA so that AP:PB = BQ:QC = CR:RD = DS:SA = 1:3. Find the ratio of the area of PQRS to that of ABCD.

F84S2. A student guesses at random at three true-false questions. What is the probability that she gets at least two correct answers?

Part II Time: 11 minutes NYCIML - Sr. A Contest 1 Fall, 1984

F84S3. In rectangle ABCD, AB = 6 and diagonal BD = 10. Circle O (with center O) is inscribed in triangle ABD and circle P (with center P) is inscribed in triangle BCD. Find the length of OP.

F84S4. The cube of a certain integer has a decimal representation consisting of ten digits, of which the two left most, as well as the rightmost, is a digit 7. Find the integer whose cube has this form.

Part III Time: 11 minutes NYCIML - Sr. A Contest 1 Fall, 1984

F84S5. Find all ordered pairs (x,y) of real numbers such that

$$3^{x^2-2xy} = 1 \quad \text{and} \quad 2\log_3 x = \log_3(y+3)\ .$$

F84S6. Find the numerical value of

$$\cos 15^\circ(\sin 75^\circ + \cos 45^\circ) + \sin 15^\circ(\cos 75^\circ - \sin 45^\circ)\ .$$

ANSWERS

1. 5:8 or equivalent
2. $\frac{1}{2}$
3. $2\sqrt{5}$
4. 1983
5. (2,1)
6. $\frac{3}{2}$

Part I   Time: 11 minutes   NYCIML - Sr. A Contest 2   Fall, 1984

F84S7. For all real non-zero numbers $f(x) = 1 - \frac{1}{x}$ and $g(x) = 1 - x$. If $h(x) = f[g(x)]$, for what value of x does $h(x) = 8$?

F84S8. Find the numerical value of $\frac{\cos 15^\circ + \sin 15^\circ}{\cos 15^\circ - \sin 15^\circ}$.

Part II   Time: 11 minutes   NYCIML - Sr. A Contest 2   Fall, 1984

F84S9. Side BC of $\triangle ABC$ is extended through C to X so that BC = CX. Similarly, side CA is extended through A to Y so that CA = AY, and side AB is extended through B to Z so that AB = BZ. Find the ratio of the area of $\triangle XYZ$ to that of $\triangle ABC$.

F84S10. Find the value of c for which the roots of $x^3 - 6x^2 - 24x + c = 0$ form an arithmetic progression.

Part III   Time: 11 minutes   NYCIML - Sr. A Contest 2   Fall, 1984

F84S11. Three ferryboats start at a terminal at noon, and go to different destinations. Ferryboat A reaches its destination after 20 minutes, boat B after 15 minutes, and boat C after 32 minutes. Upon reaching their destinations, the boats return to the terminal, then make another trip, and so on. The trip back to the terminal in each case is the same length, and takes the same time, as the trip out. What is the least number of hours after which the three ferries will again dock at the terminal simultaneously?

F84S12. In right triangle ABC, leg AC = $\sin\theta$ and leg BC = $\cos\theta$. Find the length of the longer leg if the length of the median to the hypotenuse $\overline{AB}$ is $\tan\theta$.

ANSWERS

| | | |
|---|---|---|
| 7. $x = \frac{8}{7}$ | 9. 7 or 7:1 | 11. 16 |
| 8. $\sqrt{3}$ | 10. $c = 64$ | 12. $2\sqrt{5}$ |

Part I Time: 11 minutes NYCIML - Sr. A Contest 3 Fall, 1984

F84S13. Points M, N, and P are the respective midpoints of sides $\overline{AB}$, $\overline{BC}$, and $\overline{CA}$ of $\triangle ABC$. A point X is chosen outside the plane of $\triangle ABC$. Points D, E, F are chosen such that M, N and P are respective midpoints of $\overline{XD}$, $\overline{XE}$, and $\overline{XF}$. Find the ratio of the area of triangle DEF to that of triangle ABC.

F84S14. For all real numbers x, the function f(x) satisfies

$$2f(x) + f(1-x) = x^2.$$

Find f(5).

Part II Time: 11 minutes NYCIML - Sr. A Contest 3 Fall, 1984

F84S15. In triangle ABC, AB = 5 and AC = 8. Point P is on $\overline{BC}$, and BP:PC = 3:5. Find the ratio of the radius of the circle through A, B and P to the radius of the circle through A, C and P.

F84S16. If x and y are real numbers, with $x > y$ and $xy = 1$, find the minimum possible value for

$$\frac{x^2 + y^2}{x - y}.$$

Part III Time: 11 minutes NYCIML - Sr. A Contest 3 Fall, 1984

F84S17. If $[\![x]\!]$ denotes the 'greatest integer' function, find the largest prime number p such that $\left[\!\!\left[\frac{n^2}{3}\right]\!\!\right] = p$ for some integer n.

F84S18. Square ABCD has area 1 square unit. Point P is 5 units from its center. Set S is the set of points which can be obtained by rotating point P $90^\circ$ counterclockwise about some point on or inside the square. Find the area of set S.

ANSWERS

13. 1

14. 34

15. 5:8

16. $2\sqrt{2}$

17. 5

18. 2

# New York City Interscholastic Mathematics League

Senior A Division    Contest Number 4    Fall, 1984

**Part I**  Time: 11 minutes    NYCIML - Sr. A Contest 4  Fall, 1984

F84S19. Circle O passes through vertex D of square ABCD, and is tangent to sides AB and BC. If AB = 1, the radius of circle O can be expressed as $p + q\sqrt{2}$. Find the ordered pair of rational numbers (p,q).

F84S20. Find all ordered pairs (x,y) of real numbers for which
$x^2 + xy + x = 14$ and $y^2 + xy + y = 28$.

**Part II**  Time: 11 minutes    NYCIML - Sr. A Contest 4  Fall, 1984

F84S21. In a rectangular coordinate system, a tangent from the point (24,7) to the circle whose equation is $x^2 + y^2 = 400$ has point of tangency (a,b) where $b > 0$. Find a.

F84S22. $\measuredangle ABC$ is a right angle and CB = 1. D is a point on ray $\overrightarrow{BC}$ such that DB = 3 and E is the point on ray $\overrightarrow{BA}$ such that $m\measuredangle DEC$ is maximum. Find the distance BE.

**Part III**  Time: 11 minutes    NYCIML - Sr. A Contest 4  Fall, 1984

F84S23. An ordinary pack of playing cards is shuffled, and two cards dealt face up. Find the probability that at least one of these is a spade.

F84S24. In convex quadrilateral PQRS, diagonals $\overline{PR}$ and $\overline{QS}$ intersect at T, with PT:TR = 5:4 and QT:TS = 2:5. Point X is chosen between T and S so that QT = TX, and $\overline{RX}$ is extended its own length to Y. If point Y is <u>outside</u> the quadrilateral, find the ratio of the area of triangle PSY to that of triangle QRT.

ANSWERS

19. (2,-1)
20. (2,4), (-7/3,-14/3) (both required)
21. 12
22. $\sqrt{3}$
23. $\frac{15}{34}$
24. 15:8

New York City Interscholastic Mathematics League

Senior A Division — Contest Number 5 — Fall 1984

Part I — Time: 11 minutes — NYCIML - Sr. A Contest 5 — Fall, 1984

F84S25. Point P is chosen along leg $\overline{BC}$ of right triangle ABC so that BP = PA. If leg BC = 10 and leg AC = 4, find BP.

F84S26. Five identical black socks and five identical brown socks are in a drawer. Two socks are picked at random. Find the probability that the two socks picked will match.

Part II — Time: 11 minutes — NYCIML - Sr. A Contest 5 — Fall, 1984

F84S27. The roots of $f(x) = 0$ are 2,3,7,5 and 9.
The roots of $g(x) = 0$ are 3,5,7,8, and -1.
Find all solutions of the equation $\dfrac{f(x)}{g(x)} = 0$.

F84S28. A set of distinct, non-zero real numbers is placed along the circumference of a circle. Each of the numbers is equal to the product of the two numbers adjacent to it. What is the least possible number of numbers in the set?

Part III — Time: 11 minutes — NYCIML - Sr. A Contest 5 — Fall, 1984

F84S29. In equilateral triangle ABC of edge length one, D is on $\overline{BC}$ so that $\measuredangle DAC = 45^\circ$. Find the area of triangle DAC.

F84S30. A regular 11-gon is inscribed in a circle. How many triangles are there whose three vertices are all vertices of the 11-gon and whose interiors contain the center of the circle?

ANSWERS

25. $\frac{29}{5}$ or equivalent

26. $\frac{4}{9}$

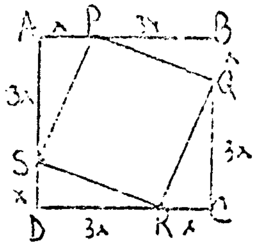
27. 2,9 both required

28. 6

29. $\frac{3-\sqrt{3}}{4}$

30. 55

SOLUTIONS.

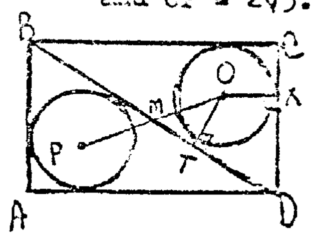


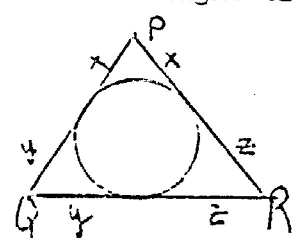

F84S1. Let $AP = x$ and $PB = 3x$. Then triangles APS, BPQ, QCR, RDS are congruent, and the sum of their areas is $6x^2$. The area of the large square is $16x^2$, so the area of the smaller square is $10x^2$, and the required ratio is $10:16 = 5:8$.

SM

F84S2. She will get all three questions correct $\frac{1}{8}$ of the time. Failing this, she will get only the first problem wrong $\frac{1}{8}$ of the time, only the second $\frac{1}{8}$ of the time, and only the third $\frac{1}{8}$ of the time. Hence she will get at least two right $\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} = \frac{1}{2}$ of the time.

F84S3. We use the lemma (see diagram) that for a circle inscribed in a triangle, $x = s - QR$, $y = s - PR$, $z = s - PQ$, where $s$ is the semiperimeter. Clearly, $BC = 8$. Using the lemma, we have $CX = OX = 2$, $XD=TD=4$. If M is the intersection of BD and OP, then $BM = MD$ (by symmetry), so $MD = 5$, and $MT = MD - TD = 1$. Hence in right triangle OMT, $OM = \sqrt{5}$ and $OP = 2\sqrt{5}$.




F84S4. The given cube is slightly less than $8 \cdot 10^9$, so the integer is slightly less than 2000. Also the rightmost digit must be a 3 (otherwise the cube would not end with a 7). The possible choices are 1993, 1983, 1973, etc. Trial and error can finish the solution. Or, we can note that

SM

$1993^3 > 1990^3 = 10^3(200-1)^3 = 10^3(8 \cdot 10^6 - 3 \cdot 4 \cdot 10^4 + 6 \cdot 10^2 - 1) >$
$10^7(800 - 12) = 788 \cdot 10^7$, so 1993 is too big.

Writing $1973^3$ as $(2 \cdot 10^3 - 3^3)^3$ and proceding similarly, we can see that this number is too small.

F84S5. We have (i) $x^2 - 2xy = 0$ and (ii) $x^2 = y + 3$.
From (i), $x(x - 2y) = 0$, so $x = 0$ or $x = 2y$. If $x = 0$, $\log_3 x$ is undefined.
If $x = 2y$, we have $4y^2 - y - 3 = 0$; $(4y + 3)(y - 1) = 0$ and
$y = -\frac{3}{4}, 1$ and $x = -\frac{3}{2}, 2$. Only the positive values will satisfy (ii).

F84S6. Expanding and rearranging the terms yields:

$(\cos 15 \sin 75 + \sin 15 \cos 75) + (\cos 15 \cos 45 - \sin 15 \sin 45) =$

$\sin(15+75) + \cos(45+15) = 1 + \frac{1}{2} = \frac{3}{2}$.

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE

Senior A Division   Contest Number 2   Fall, 1984

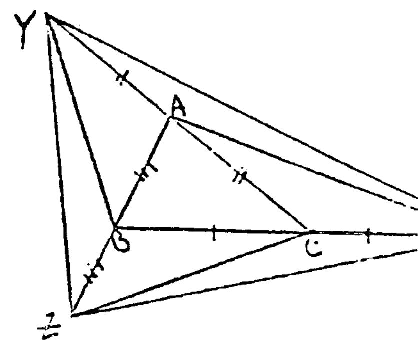
SOLUTIONS

F84S7. $h(x) = f(1-x) = 1 - \frac{1}{1-x}$. When $h(x) = 8$, we solve for x to yield $1 - \frac{1}{1-x} = 8$ so $x = \frac{8}{7}$.

F84S8. Let $N = \frac{\cos15^\circ + \sin15^\circ}{\cos15^\circ - \sin15^\circ}$. Then $N^2 = \frac{\cos^2 15^\circ + \sin^2 15^\circ + 2\cos15^\circ\sin15^\circ}{\cos^2 15^\circ + \sin^2 15^\circ - 2\cos15^\circ\sin15^\circ} =$

$$\frac{1 + \sin30^\circ}{1 - \sin30^\circ} = \frac{\frac{3}{2}}{\frac{1}{2}} = 3.$$

Since $\cos15^\circ > \frac{1}{2} > \sin15^\circ$, N is positive, and so $N = \sqrt{3}$.




F84S9. Draw XA, YB, ZC. Since BC = CX and triangles ZBC, ZCX have a common altitude, $|\Delta ZBC| = |\Delta ZCX|$ (using absolute value for area). Also, since AB = BZ, $|\Delta ACB| = |\Delta CBZ|$. Similarly, we find that $|\Delta AXC| = |\Delta AXY| = |\Delta ABC|$ and $|\Delta AYB| = |\Delta BYZ| = |\Delta ABC|$. Hence the required ratio is 7:1.

SM

F84S10. If the roots are r−d, r, and r+d, then the sum of the roots is 3r = 6, so r=2 is root. Substitution in the original equation gives:

$$8 - 24 - 48 + c = 0,$$

and c = 64.

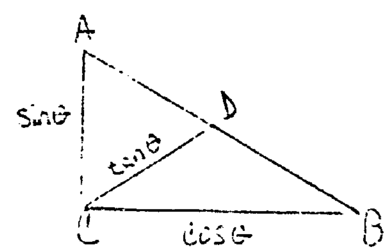
F84S11. A round trip for A takes 40 minutes. A round trip for B takes 30 minutes. A round trip for C takes 64 minutes. We need the least common multiple of these three. Since $30 = 2\cdot5\cdot3$, $40 = 2^3\cdot5$, $64 = 2^6$, the least common multiple is $2^6\cdot3\cdot5 = 64\cdot15$ minutes, or $64\cdot\frac{1}{4} = 16$ hours.

MS

F84S12. From the Pythagorean Theorem AB = 1. Also, the median to the hypotenuse is half as long as the hypotenuse. Therefore: $\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{1}{2}$. From this equation, $\cos\theta$ is clearly the longer leg, and by the Pythagorean Theorem:

$$\left(\frac{\cos\theta}{2}\right)^2 + \cos^2\theta = 1$$ and solving yields

$$\cos\theta = \frac{2\sqrt{5}}{5}.$$



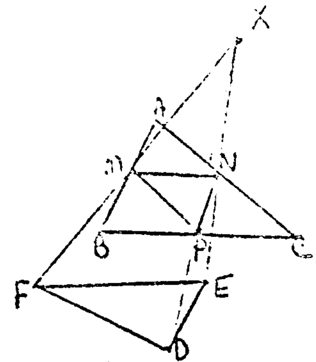
NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE

Senior A Division                Contest Number 3                Fall, 1984

SOLUTIONS

F84S13. A line connecting two midpoints of a triangle is equal to half the third side. Hence $MP = \frac{1}{2}FD$ and $NP = \frac{1}{2}AC$, so $AC = DF$. Similarly, $BC = EF$ and $AB = DE$, so that the required ratio is 1.

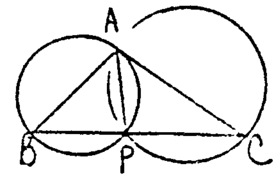
SM




F84S14. Letting $x = 5$, we have $2f(5) + f(-4) = 25$. Letting $x = -4$, we have $2f(-4) + f(5) = 16$. These are two equations in two unknowns. Solving, we find $f(5) = \frac{34}{3}$, $f(-4) = \frac{7}{3}$. In general, $f(x) = \frac{x^2+2x-1}{3}$.

F84S15. The length of a chord subtended by an inscribed angle $\theta$ in a circle of radius $x$ is $2x\sin\theta$. Let $\measuredangle ABC = \beta$, $\measuredangle ACB = \gamma$, and the radii of the circles be $r$ and $R$ (the circle of radius $r$ is through A, B, and P). Then $AP = 2r\sin\beta = 2R\sin\gamma$, so $r:R = \sin\gamma : \sin\beta$
$= 5:8$
(by the law of sines).

SM

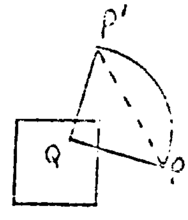



F84S16. We have $\frac{x^2+y^2}{x-y} = \frac{x^2-2xy+y^2+2xy}{x-y} = \frac{(x-y)^2+2}{x-y} = (x-y) + \frac{2}{x-y}$.

Let $z = x-y$. Then $z > 0$, and $z\cdot\frac{2}{z} = 2$, a constant, so $z + \frac{2}{z}$ is minimal when $z = \frac{2}{z}$, or $z = \sqrt{2}$. This gives $x = \frac{\sqrt{6}+\sqrt{2}}{2}$, $y = \frac{\sqrt{6}-\sqrt{2}}{2}$, and $\frac{x^2+y^2}{x-y} = 2\sqrt{2}$. Except to check that both are positive, it is not necessary to solve explicitly for $x$ and $y$. We have used the fact that if the product of two quantities is constant, their sum is minimal when the quantities are equal. This is a consequence, for instance, of the simplest case of the arithmetic mean-geometric mean inequality. See, for instance, Beckenbach & Bellman, An Introduction to Inequaltities.

F84S17. We need only consider $n > 0$. Let $n = 3k+\ell$ where $\ell = 0,1,2$ and $k > 0$.
If $\ell = 0$, $n^2 = 9k^2$, $[\![\frac{n^2}{3}]\!] = 3k^2$, prime only for $k = 1$.
If $\ell = 1$, $[\![\frac{n^2}{3}]\!] = 3k^2 + 2k = k(3k+2)$, prime only for $k = 1$.
If $\ell = 2$, $[\![\frac{n^2}{3}]\!] = 3k^2 + 4k + 1 = (3k+1)(k+1)$, which cannot be prime.
Letting $n = 3\cdot 1 + 1 = 4$, $[\![\frac{n^2}{3}]\!] = 5$.

F84S18. Let P' be the image of P when rotated about a typical point Q. Triangle PP'Q is always right and isosceles, so $\measuredangle QPP' = 45^0$, and $PP' = QP\sqrt{2}$. Hence we can obtain P' from Q by rotating $45^o$ about P and dilating the resulting figure (performing a homothecy) about P by a factor of $\sqrt{2}$. This means that the locus of P' is another square, whose side is $\sqrt{2}$, and whose



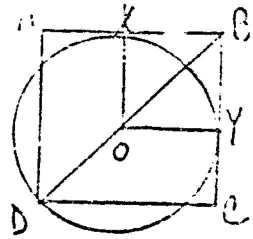
NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE

Senior A Division | Contest Number 4 | Fall, 1984

## SOLUTIONS

F84S19. SM

Let circle O be tangent to AB at X and to BC at Y. By symmetry, point O is on diagonal BD. If OD = OX = OY = r, then OXBY is a square, so OB = $r\sqrt{2}$. Thus $OD + OB = r + r\sqrt{2} = OB = \sqrt{2}$, so $r = \frac{\sqrt{2}}{1+\sqrt{2}} = -2(1-\sqrt{2}) = 2-\sqrt{2}$, and $(p,q) = (2,-1)$.




F84S20. MS

Adding, we have $x^2 + 2xy + y^2 + x + y = 42$, $(x+y)^2 + (x+y) - 42 = 0$.
Let $x + y = z$. Then $z^2 + z - 42 = 0$, or $z = -7, 6$.
Subtracting the original equations, we have $x^2 - y^2 + x - y =$

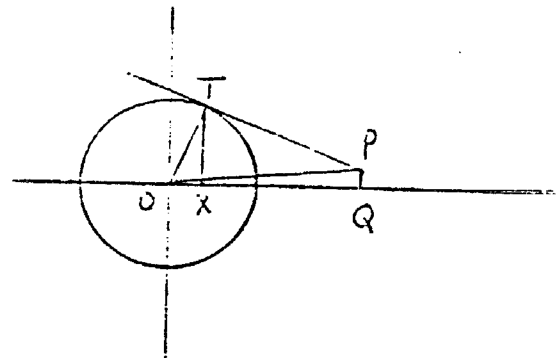
$$(x+y)(x-y) + (x-y) = (x+y+1)(x-y) = -14,$$

so $x - y = \frac{7}{3}$, $x + y = -7$, $(x,y) = (-7/3, -14/3)$

or $x - y = -2$, $x + y = 6$, $(x,y) = (2,4)$

F84S21. MS

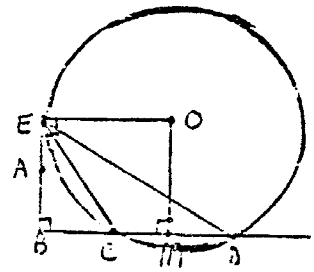
Clearly OT = 20. Since OQ = 24, QP = 7, we know OP = 25. Then, if $\measuredangle POQ = \theta$, $\measuredangle TOP = \phi$, we can write $\measuredangle TOQ = (\theta + \phi)$ and $OX = a = OT\cos(\theta + \phi)$.
Now $\cos\theta = \frac{OQ}{OP} = \frac{24}{25}$, $\cos\phi = \frac{OT}{OP} = \frac{20}{25}$,
so $\cos(\theta + \phi) = \frac{24}{25}\cdot\frac{20}{25} - \frac{7}{25}\cdot\frac{15}{25} = \frac{375}{625} = \frac{3}{5}$,
and $OX = 20\cdot\frac{3}{5} = 12$.




F84S22.

If we draw the circle through D and C which is tangent to AB at E, then angle DEC is $\frac{1}{2}$ arc CD. For any other point E on AB, $\measuredangle DEC$ is less than $\frac{1}{2}\widehat{CD}$, since other points are outside the circle. Hence point E is the point we want.
To find BE, let O be the center of the circle, and M the midpoint of CD. Then $CM \perp CD$, $OE \perp BE$, and $OE = BM = 2$. Then $EB = MO$, and $EB^2 = MO^2 = OD^2 - MD^2 =$

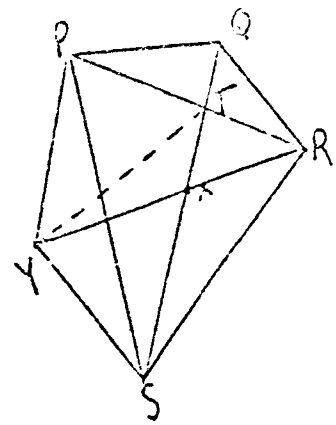
$$OE^2 - MD^2 = 4 - 1 = 3.$$




F84S23. The probability of <u>neither</u> being a spade is $\frac{39}{52}\cdot\frac{38}{51} = \frac{19}{34}$. Hence the probability of getting at least one spade is $1 - \frac{19}{34} = \frac{15}{34}$.

F84S24. We compute the ratios of areas of triangles with equal altitudes by compari their bases. Using absolute value for area, let $|QRT| = A$. Then $|TRX| = A$, $|RXS| = \frac{3}{2}A = |XSY|$, $|YTX| = |TRX| = A$ (all from ratios along line QS).
Also $|PTS| = \frac{25}{3}A$.
From ratios along PR: $|PYT| = \frac{3}{2}A$. Thus $|PTSY| = |PTY| + |TYX| + |YXS| =$
and $|PYS| = |PTSY| - |PTS| = 5A - \frac{25}{3}A = \underline{15}A$.

[illegible]4: diagram

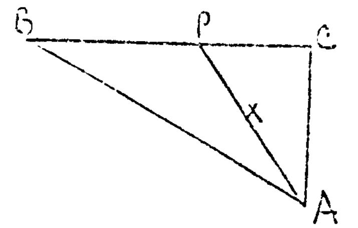


NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE

Senior A Division | Contest Number 5 | Fall, 1984

SOLUTIONS

F84S25. If PA = x = PB then PC = 10 − x, and $PC^2 + CA^2 = PA^2$, or $(10-x)^2 + 4^2 = x^2$,
MS
$100 - 20x + x^2 + 16 = x^2$, $116 - 20x = 0$, $x = \frac{116}{20} = \frac{29}{5}$.




F84S26. After the first sock is picked, the chance of the second sock matching is 4/9.

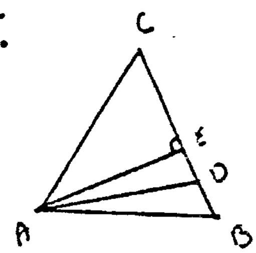
F84S27. $\frac{f(x)}{g(x)} = 0$ if and only if $f(x) = 0$ while $g(x) \neq 0$. This happens for 2 and 9.

F84S28. Suppose there are n numbers in the set. Clearly n is more than 2. We will first show that in fact n cannot be 3,4, or 5. For example, if n = 3, the numbers can be represented by a,b, and b/a, so that b/a = ab. Hence $|a| = |b| = |b/a| = 1$. But then at least two of the three numbers must be equal, contrary to the assumptions of the problem. Similar arguments will show that n cannot be 4 or 5.
Next we can show that n cannot be greater than 6. Indeed, if the first two numbers are a and b, then the next five are b/a, 1/a, 1/b, a/b, and a, so that the seventh number is equal to the first. Hence n = 6.
The sequence 2,3,3/2,1/2,1/3,2/3 (among others) will in fact satisfy the conditions of the problem.

F84S29. Drop altitude AE onto BC. Then AD is the bisector of angle EAB, so BD:DE = AB:AE = $2:\sqrt{3}$. This leads to DB = $1/(2+\sqrt{3})$, and DC =

$$\frac{1+\sqrt{3}}{2+\sqrt{3}}$$

. Then the area of triangle ACD is (1/2)(DC)(AE) = $(3-\sqrt{3})/4$.




F84S30. Label any vertex of the 11-gon as point A. Then we can label the next five vertices going clockwise around the circle as B1, B2, B3, B4, B5, and the five vertices going counter-clockwise as C1, C2, C3, C4, and C5. Let us first find out how many of the triangles described have a vertex at A. Clearly, of the other two vertices of such a triangle, one must be some Bi and another some Cj for $1 \le i, j \le 5$. And in fact, it is not hard to see that $i + j \geq 6$ if the triangle is to contain the center of the circle. Now it is not hard to count the triangles: each corresponds to solution of the above inequality for i and j natural numbers less than 6. There is 1 solution if i = 1, two if i = 2, and so on. Hence there are 1+2+3+4+5=15 triangles with one vertex at A.
Letting each vertex of the 11-gon play the role of A, we now have $11 \cdot 15 = 165$ triangles. But we have then counted each triangle three times (having let each vertex of each triangle play the role of A), so that the number of triangles is actually 165/3 = 55.
This argument can be generalized for a regular polygon with an odd number of sides.