New York City
Interscholastic Mathematics League

Junior Division — Contest Number 1 — Fall, 1984

Part I Time: 11 minutes — NYCIML - Jr. Contest 1 — Fall, 1984

F84J1. If $2x + y - z = 12$ and $x + 2y - z = 7$, find the numerical value of $x - y$.

F84J2. Find the smallest positive integer which leaves a remainder of 2 when divided by 3 or by 7 and a remainder of 9 when divided by 11.

Part II Time: 11 minutes — NYCIML - Jr. Contest 1 — Fall, 1984

F84J3. Five men can plow a square field whose side is 60 feet in 4 hours. At the same rate of work, in how many hours could 10 men plow a square field whose side is 180 feet?

F84J4. In square ABCD, E is the midpoint of $\overline{AB}$. $\overline{AC}$ meets $\overline{DE}$ in F. Find the value of the ratio DF:FE.

Part III Time: 11 minutes — NYCIML - Jr. Contest 1 — Fall, 1984

F84J5. Find the smallest real number x such that

$$|x-1| - 2|x+3| + x + 7 = 0.$$

F84J6. In right triangle ABC, leg $AC = 2\sqrt{3}$ and leg $BC = 6$. Points P, Q, and R are chosen on line segments $\overline{AC}$, $\overline{BC}$, and $\overline{AB}$ respectively such that $\overline{PQ}$ is parallel to $\overline{AB}$ and ΔPQR is an equilateral triangle. Find the length of $\overline{PQ}$.

ANSWERS

1. 5	3. 18 or 18 hours	5. −7
2. 86	4. 2:1	6. $\frac{4\sqrt{3}}{3}$ or equivalent

New York City
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Junior Division Contest Number 2 Fall, 1984

Part I Time: 11 minutes NYCIML - Jr.Contest 2 Fall, 1984

F84J7. A man can dig a rectangular ditch 3 feet wide, 2 feet long and 4 feet deep in 10 minutes. If two men work together at the same rate, how many minutes will it take them to dig a ditch 6 feet wide, 4 feet long and 8 feet deep?

F84J8. A pile of pennies, dimes and half dollars, 100 coins in all, is worth \$5.00. How many dimes are in the pile?

Part II Time: 11 minutes NYCIML - Jr. Contest 2 Fall, 1984

F84J9. The lengths of the sides of a right triangle are all integers. The lengths of the legs are a and b, and of the hypotenuse is c. If a is a prime number, find the numerical value of $c - b$.

F84J10. Circle I and Circle II intersect at points M and N. A line through M intersects Circle I again at A and Circle II again at B, with M between A and B. A line through N intersects Circle I again at C and Circle II again at D, with N between C and D. If $m\angle CAB = 84^\circ$, find the degree-measure of $\angle ABD$.

Part III Time: 11 minutes NYCIML - Jr. Contest 2 Fall, 1984

F84J11. If $x^2 - y^2 = 2xy$ and x and y are both positive, find the ratio $\frac{x}{y}$.

F84J12. Lines L_1 and L_2 are parallel and lie one unit apart. Point P is fixed on L_1 and point X varies along L_2. Line $\overleftrightarrow{XY}$ is perpendicular to $\overline{PX}$, with Y on L_1. Find the minimum possible length of $\overline{PY}$.

ANSWERS

7. 40 or 40 minutes
8. 39
9. 1
10. 95°
11. $1 + \sqrt{2}$
12. 2

New York City
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Junior Division | Contest Number 3 | Fall, 1984

Part I Time: 11 minutes NYCIML - Jr. Contest 3 Fall, 1984

F84J13. For what values of m does the equation

$$(m-1)x^2 - 4x + (m+2) = 0$$

have two equal roots?

F84J14. A rectangular box is 6 inches long, 8 inches wide and 5 inches high. A straight metal rod of negligible thickness longer than 10 inches can be fit into the box. What is the largest possible inch-length of such a rod (leave answer in radical form)?

Part II Time: 11 minutes NYCIML - Jr. Contest 3 Fall, 1984

F84J15. Of two identical barrels, one is $\frac{1}{2}$ full and one is $\frac{2}{3}$ full. One quarter of the liquid in the second barrel is poured into the first. The first barrel now contains 25 more gallons of liquid than the second. Find the capacity in gallons of one of the barrels.

F84J16. A sphere whose center is at point O has radius 2. From point P (outside the sphere), tangent lines are drawn to the sphere. The points of tangency form a circle on the surface of the sphere. If OP = 4, find the area of this circle.

Part III Time: 11 minutes NYCIML - Jr. Contest 3 Fall, 1984

F84J17. The sum of n consecutive integers (some of which may be negative) with $n \geq 2$ equals 175. For how many values of n is this possible?

F84J18. Chord $\overline{AB}$ of a circle is extended through B to point D, and tangent $\overline{DC}$ is drawn to the circle. If AB = CD and the area of triangle BCD is $2\sqrt{5}$ square centimeters, then the area of triangle ABC is $a + b\sqrt{5}$ square centimeters. Find the ordered pair (a,b).

ANSWERS

13. 2,-3

15. 150 gallons

17. ~~5~~ 11

14. [illegible]

16. [illegible]

18. [illegible]

Junior Division Contest Number 1 Fall, 1984

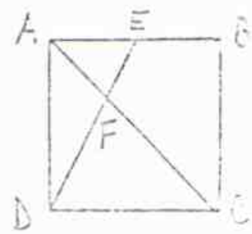
SOLUTIONS

F84J1. $2x + y + z = 12$
$x + 2y + z = 7$
Subtracting, $x - y = 5$.

F84J2. If the number we want is N, then $N - 2$ is a multiple of 3 or 7, hence a multiple of 21. Since $N - 9 = N - 2 - 7$ is a multiple of 11,
SM we need to find a multiple of 21 which is 7 more than a multiple of 11. Checking multiples of 21 shows 84 is the first such, so $N = 86$. For more general methods, see the Chinese Remainder Theorem in any number theory book.

F84J3. A side of the second field is three times a side of the first, so the area of the second field is nine times the area of the first. It would take 5 men $4 \times 9 = 36$ hours to plow this second field, so it will take 10 men 18 hours.

F84J4. Since $\triangle AEF \sim \triangle DFC$, $DC:AE = FD:FE = 2:1$.



F84J5. If $x > 1$, we have $(x-1) - 2(x+3) + x + 7 = 0$ identically.
If $x \leq 1$ and $x \geq -3$, we have $1 - x - 2(x+3) + x + 7 = 0$, for which $x = 1$ is the only solution.
If $x < -3$, we have $1 - x + 2(x+3) + x + 7 = 0$, or $x = -7$.

F84J6. Note that $m\angle APR = m\angle CPQ = m\angle RPA = 60^\circ$. Hence, if $PQ = x$, then $PR = PA = x$, while $PC = x/2$ (triangle CPE is a 30-60-90 triangle). Hence $CP + PA = 3x/2 = CA = 2\sqrt{3}$, and $x = 4\sqrt{3}/3$.

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE

Junior Division Contest Number 2 Fall, 1984

SOLUTIONS

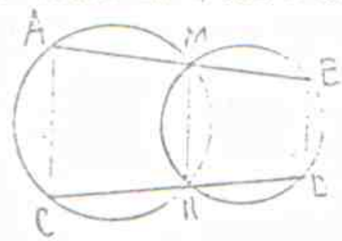
F84J7. The dimensions of the second ditch are twice as big as those of the first, so its volume is 8 times that of the first. Hence it will take two men four times as long to dig, or 40 minutes. Compare F84J5.

F84J8. If the number of pennies, dimes and half dollars is represented by x,y, and z respectively, we have:
$$x + y + z = 100$$
$$x + 10y + 50z = 500.$$
Subtracting, $9y + 49z = 400$, $y = \dfrac{400 - 49z}{9} = 44 - 5z + \dfrac{4 - 4z}{9}$.

Thus 9 must divide $4 - 4z$. Also, $z < 10$. Only $z = 1$ works. Then $9y = 351$ and $y = 39$.

F84J9. We have $a^2 = c^2 - b^2 = (c+b)(c-b)$. Since a is prime, a^2 can factor
MS only as $1 \cdot a^2$, $a \cdot a$, or $a^2 \cdot 1$. Since $c-b < c+b$, $c-b$ must be 1.

F84J10. If a quadrilateral is inscribed in a circle, then its opposite angles are supplementary. Hence if $\measuredangle CAB = 84^\circ$, $\angle CNM = 96^\circ$. Then $\measuredangle MND = 84^\circ$ and $\measuredangle MBD = 96^\circ$.



F84J11. $x^2 - y^2 = 2xy$, so $\dfrac{x^2}{y^2} - 1 = \dfrac{2x}{y}$. Letting $\dfrac{x}{y} = r$,

we have $r^2 - 1 = 2r$; $r^2 - 2r - 1 = 0$; $r = \dfrac{2 \pm \sqrt{8}}{2} = 1 \pm \sqrt{2}$.

We reject the negative value.

F84J12. For any position of X, $\overline{PY}$ is the diameter of a circle intersecting L_2 at X. We wish to find the diameter of the smallest circle through P which still intersects L_2. Clearly, this will be the circle tangent to L_2 through P, which has a diameter of 2 (see diagram).

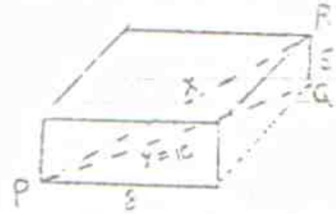
NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE

Junior Division | Contest Number 3 | Fall, 1984

SOLUTIONS

F84J13. The discriminant is 0 when $16 - 4(m-1)(m+2) = 0$; $m^2 + m - 6 = 0$, $m = 2, -3$.

F84J14. First we look at the bottom of the box. It is a rectangle, and the Pythagorean Theorem shows that its diagonal is 10 inches long: $6^2 + 8^2 = y^2$; $100 = y^2$; $10 = y$. To find the required length, which is a diagonal that goes through the box, we use the Pythagorean Theorem again, this time in PQR (see diagram):
$x^2 = PQ^2 + QR^2 = 10^2 + 5^2 = 125$; $x = \sqrt{125} = 5\sqrt{5}$.



F84J15. If the capacity is x gallons, then

GEML

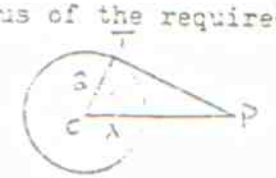
$$25 = \frac{1}{2}x + \frac{1}{4}\cdot\frac{2}{3}x - \frac{2}{3}x - \frac{1}{4}\cdot\frac{2}{3}x$$
$$= \frac{1}{2}x + \frac{1}{6}x - \frac{2}{3}x + \frac{1}{6}x$$
$$= \frac{1}{2}x - \frac{1}{3}x = \frac{1}{6}x.$$

Hence $x = 150$ gallons.

F84J16. Pass a plane through CP to form the cross section shown. Then $PT = \sqrt{12} = 2\sqrt{3}$, and $TX \cdot CP = CT \cdot PT$ (for example, by calculating the area of $\triangle CTP$ in two different ways), so $TX = \frac{4\sqrt{3}}{4} = \sqrt{3}$.

SM

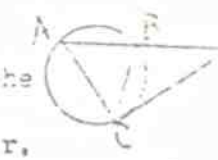
But TX is the radius of the required circle, so the area is 3π.



F84J17. Suppose the integers are $a+1, a+2, \ldots a+n$. Their sum is $n\cdot a + \frac{n(n+1)}{2} = n\left[a + \frac{n+1}{2}\right]$. Since this product is $175 = 7\cdot 5^2$, the possible values for n are 1,5,7,25,35,175. Since these are all odd, they each give a corresponding value for a, except $n = 1$, which is excluded by the conditions of the problem.

JR

F84J18. Let $AB = CD = x$, $BD = y$. Then $x^2 = y(x+y)$ or $x^2 - y^2 = xy$. We can solve this for the ratio $\frac{x}{y}$, which is also the ratio of the area of $\triangle ABC$ to that of $\triangle BCD$. We have $\left(\frac{x}{y}\right)^2 - 1 = \frac{x}{y}$. If $\frac{x}{y} = r$, this is $r^2 - r - 1 = 0$, and $\frac{x}{y} = r = \frac{1+\sqrt{5}}{2}$ (discarding the negative root).



Then $\triangle ADC = 2\sqrt{5}\left(\frac{1+\sqrt{5}}{2}\right) = 5 + \sqrt{5}$.